ON LENGTH DENSITIES

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Abstract. For a commutative cancellative monoid M, we introduce the notion of the *length density* of both a nonunit $x \in M$, denoted $\mathrm{LD}(x)$, and the entire monoid M, denoted $\mathrm{LD}(M)$. This invariant is related to three widely studied invariants in the theory of non-unit factorizations, $L(x)$, $\ell(x)$, and $\rho(x)$. We consider some general properties of $\mathrm{LD}(x)$ and $\mathrm{LD}(M)$ and give a wide variety of examples using numerical semigroups, Puiseux monoids, and Krull monoids. While we give an example of a monoid M with irrational length density, we show that if M is finitely generated, then $\mathrm{LD}(M)$ is rational and there is a nonunit element $x \in M$ with $\mathrm{LD}(M) = \mathrm{LD}(x)$ (such a monoid is said to have accepted length density). While it is well-known that the much studied asymptotic versions of $L(x)$, $\ell(x)$ and $\rho(x)$ (denoted $\overline{L}(x)$, $\overline{\ell}(x)$, and $\overline{\rho}(x)$) always exist, we show the somewhat surprising result that $\overline{\mathrm{LD}}(x) = \lim_{n\to\infty} \mathrm{LD}(x^n)$ may not exist. We also give some finiteness conditions on M that force the existence of $\overline{\mathrm{LD}}(x)$.

1. Introduction

A commutative cancellative monoid M with set of irreducible elements (or atoms) $\mathcal{A}(M)$ is called *atomic* if for each nonunit $x \in M$ there are $x_1, \ldots, x_k \in \mathcal{A}(M)$ such that $x = x_1 \cdots x_k$. For such an x, set

$$\mathsf{L}(x) = \{k \in \mathbb{N} \mid \text{there exist atoms } x_1, \ldots, x_k \text{ such that } x = x_1 \cdots x_k\}. \tag{1}$$

The set $\mathsf{L}(x)$ is known as the *set of lengths* of $x \in M$ (see [31]), and its study over the past 60 years has been the principal focus of non-unique factorization theory (the monograph [33] is a good general source on this topic). If for each nonunit $x \in M$ we have that $|\mathsf{L}(x)| < \infty$, then M is called a *bounded factorization monoid* (or *BF-monoid*). Consider several of the fundamental results for BF-monoids. For instance, set

$$L(x) = \max \mathsf{L}(x), \quad \ell(x) = \min \mathsf{L}(x), \quad \rho(x) = \frac{L(x)}{\ell(x)}, \quad \text{and} \quad \rho(M) = \sup\{\rho(x) \mid x \in M\}. \tag{2}$$

The constant $\rho(x)$ is known as the *elasticity* of x in M and the constant $\rho(M)$ as the *elasticity of M*. In [3] the authors show that if M is the multiplicative monoid of a Krull domain with finite divisor class group, then $\rho(M)$ is rational and moreover there exists a nonunit $x \in M$ so that $\rho(M) = \rho(x)$ (such a monoid is said to have *accepted elasticity*). If we further set

$$\overline{L}(x) = \lim_{n\to\infty} \frac{L(x^n)}{n} \quad \text{and} \quad \overline{\ell}(x) = \lim_{n\to\infty} \frac{\ell(x^n)}{n}, \tag{3}$$

then Anderson and Pruis show in [6] that

(i) both the limits $\overline{L}(x)$ and $\overline{\ell}(x)$ exist (although $\overline{L}(x)$ may be infinite);

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(ii) if α and $\beta \in [0, \infty]$ with $0 \leq \alpha \leq 1 \leq \beta \leq \infty$, then there is an integral domain R and an irreducible $x \in R$ with $\overline{\ell}(x) = \alpha$ and $\overline{L}(x) = \beta$.

These are but a few of the numerous constants that have been attached to M and x to better describe their factorization properties. The above constants are rather "coarse" in the sense that they merely describe the extreme values in $\mathsf{L}(x)$. The purpose of this note is to introduce a new constant, finer in nature, which describes not just extremes, but the entire length set.

Definitions 1.1. Let M be a commutative cancellative atomic BF-monoid with set of units $M^\times$. Define a function $\mathsf{L}^\Delta : M \to \mathbb{N}_0$ via

$$\mathsf{L}^\Delta(x) = L(x) - \ell(x)$$

where we define $\mathsf{L}^\Delta(x) = 0$ if $x \in M^\times$. We define the *length ideal* of M, denoted M^{LI} as the set of elements with nonzero image under L^Δ. For $x \in M^{LI}$ set

$$\mathrm{LD}(x) = \frac{|\mathsf{L}(x)| - 1}{\mathsf{L}^\Delta(x)},$$

which we call the *length density* of x. Moreover, set

$$\mathrm{LD}(M) = \inf\{\mathrm{LD}(x) \mid x \in M^{LI}\},$$

which we call the *length density* of M. If there is an $x \in M^{LI}$ such that $\mathrm{LD}(M) = \mathrm{LD}(x)$, then we say that the length density of M is *accepted*. Set

$$\overline{\mathrm{LD}}(x) = \lim_{n \to \infty} \mathrm{LD}(x^n)$$

to be the *asymptotic length density of* x, provided this limit exists.

Notice that M is half-factorial if and only if M^{LI} is empty; we henceforth exclude such monoids from consideration. Clearly M^{LI} is a semigroup ideal, as $M^{LI}M \subseteq M^{LI}$. Under the hypothesis of Definitions 1.1, each $\mathrm{LD}(x)$ is a rational number in the interval $(0, 1]$ and so $0 \leq \mathrm{LD}(M) \leq 1$. In particular, if M has accepted length density, then the length density of M must be a rational number. Before considering further the inequality $0 \leq \mathrm{LD}(M) \leq 1$, we require some additional notation. We will call a subset $S \subseteq \mathbb{N}$ an *interval* if $S = [\min S, \max S] \cap \mathbb{N}$. If $\mathsf{L}(x) = \{n_1, n_2, \ldots, n_k\}$, where $n_1 < n_2 < \cdots < n_k$, then the *delta set* of x and M are defined as

$$\Delta(x) = \{n_{i+1} - n_i \mid 1 \leq i < k\} \quad \text{and} \quad \Delta(M) = \bigcup_{x \in M} \Delta(x),$$

respectively. There is a wealth of literature concerning the delta set of various types of commutative cancellative monoids: Krull monoids [9, 16, 20, 34, 36, 38]; numerical monoids [14, 26, 27, 35]; Puiseux monoids [15, 39]; and arithmetic congruence monoids [8, 10]. We assume the reader has a working knowledge of the terminology and basic properties of these types of monoids.

We break our remaining work into 3 sections. In Section 2, we review some basic properties of length density and in Proposition 2.3 offer bounds on $\mathrm{LD}(x)$ and $\mathrm{LD}(M)$. We consider when the extreme values in these bounds are met and offer a wide array of examples of such behavior. In Section 3, Proposition 3.1 allows one to construct monoids with both arbitrary elasticity and length density. This construction can be used to construct monoids with irrational length density. We follow this in Theorem 3.4 by arguing that a finitely generated monoid has rational length density which is accepted. We close Section 3 with a brief discussion of the computation of length density for block monoids, and in turn for general Krull monoids. Section 4 begins with Example 4.1 which illustrates that the asymptotic length density of an element may not exist; this is in stark constrast to the previously mentioned results for $\overline{\ell}(x)$ and $\overline{L}(x)$ (as well as the asymptotic elasticity defined by $\overline{\rho}(x) = \overline{L}(x)/\overline{\ell}(x)$). We then give conditions in Theorem 4.2 on an atomic monoid which guarantee the existence of $\overline{\mathrm{LD}}(x)$, and note that finitely generated monoids and Krull monoids with finite divisor class group satisfy these conditions. Our work is generally self contained; we direct the reader to [33] for any undefined terminology or background.

2. Basic Ideas and Bounds on the Length Density

We open by considering the largest value that $\mathrm{LD}(x)$ can attain.

Proposition 2.1. *Let M be a commutative cancellative atomic monoid and $x \in M^{LI}$. The following statements are equivalent.*

(1) $\mathrm{LD}(x) = 1$.
(2) $\mathsf{L}(x)$ *is an interval.*
(3) $\Delta(x) = \{1\}$.

If all the elements of M^{LI} satisfy any of these conditions, then M necessarily has accepted length density.

Example 2.2. If $\mathrm{LD}(M) = 1$, then all elements of its length ideal satisfy the conditions of Proposition 2.1 and $\Delta(M) = \{1\}$. In the general scheme of factorization theory, such monoids have appeared in the literature, but have not been widely studied. Hence, we offer several examples.

(1) A numerical monoid is any cofinite additive submonoid of $\mathbb{N}_0$. Let M be a numerical monoid generated by an interval of integers (i.e., $M = \langle n, n+1, n+2, \ldots, n+k\rangle$ where $k \leq n-1$). By [14, Theorem 3.9], $\Delta(M) = \{1\}$. This relationship does not work conversely. Additionally, let $r_1 < r_2 < ... < r_k$ be natural numbers with $\gcd(r_1, \ldots, r_k) = 1$ and set $n > r_k^2$. Then, by [22, Corollary 5.7], the delta set of the shifted numerical monoid $\langle n, n + r_1, n + r_2, ..., n + r_k\rangle$ is $\{1\}$.
(2) Let M be a Krull monoid with finite divisor class group G such that each divisor class of G contains a prime divisor. By [30, Corollary 2.3.5], the only two cases where $\Delta(M) = \{1\}$ occur are when $G = \mathbb{Z}_3$ or $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus, for instance, the elements of an algebraic ring of integers with class number 3 will satisfy Proposition 2.1. A simplier construction is also possible. Let $M = \{(x_1, x_2, x_3) \in \mathbb{N}_0^3 \mid x_1 + 2x_2 = 3x_3\}$. By [18, Theorem 1.3], M is a Krull monoid under addition with class group $\mathbb{Z}_3$ which has $\Delta(M) = \{1\}$.
(3) If a and b are positive integers with $a \leq b$ and $a^2 \equiv a \bmod b$, then the set

$$M_{a,b} = \{1\} \cup \{x \in \mathbb{N} \mid x \equiv a \bmod b\}$$

is a multiplicative monoid known as an *arithmetical congruence monoid* (or *ACM*). If $\gcd(a, b) = p^\alpha$ for p a prime, then we call M *local*. If M is local, then let β be minimal with p^β in M (note that $\beta \geq \alpha$). By [8, Theorem 3.1], if either $\alpha = \beta = 1$ or if $\alpha < \beta \leq 2\alpha$, then $\Delta(M) = \{1\}$. So, for instance, all the elements of both $M_{4,6}$ and

$M_{96,160}$ satisfy Proposition 2.1. Thus, by [11, Theorem 1], both $M_{4,6}$ and $M_{96,160}$ are monoids with accepted length density but not accepted elasticity.

If $\gcd(a,b) > 1$ is composite, then we call M *global*. Among the various conditions given in [8] which force a global ACM to have $\Delta(M) = \{1\}$ is Corollary 4.5 which shows that this is the case when $a = b$. So while it is easy to verify that $\Delta(M) = \emptyset$ when $M = p\mathbb{N} \cup \{1\}$ for p a prime, $\Delta(M) = \{1\}$ for monoids like $6\mathbb{N} \cup \{1\}$ and $9\mathbb{N} \cup \{1\}$. Moreover, in the case where $a = b$ is composite and not a power of a prime, then it is easy to verify that $\rho(M) = \infty$. Thus $0 < \mathrm{LD}(M)$ does not imply that $\rho(M) < \infty$.

(4) The last example motivates another class of monoids. An atomic monoid M is called *bifurcus* if every nonzero, nonunit of M can be factored into a product of two irreducibles. By [1, Theorem 1.1 (3)], for any $x \in M^{LI}$, $\mathsf{L}(x) = \{2, 3, \ldots, L(x)\}$, so by definition, $\Delta(M) = \{1\}$. Various examples of rings that satisfy this condition can be found in [1, 7], including:
   (a) $n\mathbb{Z}$ for n not a prime power (bifurcus ring, without identity);
   (b) $(m\mathbb{Z}) \times (n\mathbb{Z})$ for m, n each greater than 1 (bifurcus ring, without identity); and
   (c) the subring of $n \times n$ matrices consisting of matrices where all entries are identical integers and n is not a prime power (bifurcus ring, without identity).

We now provide a more general proposition.

Proposition 2.3. *If $x \in M^{LI}$, then*

$$\frac{1}{\max \Delta(x)} \leq \mathrm{LD}(x) \leq \frac{1}{\min \Delta(x)}, \tag{4}$$

with equality on either side implying $|\Delta(x)| = 1$, which in turn implies equality on both sides. Furthermore,

$$\frac{1}{\sup \Delta(M)} \leq \mathrm{LD}(M) \leq \frac{1}{\min \Delta(M)}, \tag{5}$$

with equality on the right side implying $|\Delta(M)| = 1$, which in turn implies equality on both sides.

Proof. Suppose $\mathsf{L}(x) = \{l_0, l_1, \ldots, l_k\}$ with $l_0 < l_1 < \cdots < l_k$. Then

$$\begin{aligned} L(x) - \ell(x) &= l_k - l_0 \\ &= (l_k - l_{k-1}) + (l_{k-1} - l_{k-2}) + \cdots + (l_1 - l_0) \\ &= \textstyle\sum_{i=1}^{k} l_i - l_{i-1}. \end{aligned}$$

Since each summand is an element of $\Delta(x)$,

$$k \min \Delta(x) \leq \sum_{i=1}^{k} l_i - l_{i-1} \leq k \max \Delta(x).$$

Dividing throughout by $|\mathsf{L}(x)| - 1 = k$, we get

$$\min \Delta(x) \leq \frac{L(x) - l(x)}{|\mathsf{L}(x)| - 1} \leq \max \Delta(x).$$

Taking reciprocals gives the double inequality in the theorem statement. If either inequality is actually equality, then all summands are equal, and hence $|\Delta(x)| = 1$. The last inequalities now easily follow. □

Immediately we obtain the following.

Corollary 2.4. *Let M be an atomic monoid. If $\Delta(M) = \{d\}$ for some positive integer d, then $\mathrm{LD}(x) = \frac{1}{d}$ and consequently $\overline{\mathrm{LD}}(x) = \frac{1}{d}$ for all $x \in M^{LI}$. It follows that $\mathrm{LD}(M) = \frac{1}{d}$ and that the length density of M is accepted.*

Example 2.5. We offer some concrete examples to illustrate the last two results.

(1) Let M be a numerical monoid generated by an arithmetic sequence of integers (i.e., $M = \langle a, a+d, \ldots, a+kd\rangle$ where $\gcd(a, d) = 1$ and $k < a$). By [14, Theorem 3.9], $\Delta(M) = \{d\}$, and by Corollary 2.4, $\mathrm{LD}(x) = \overline{\mathrm{LD}}(x) = \frac{1}{d}$ for all nonunits $x \in M$. As such, $\mathrm{LD}(M) = \frac{1}{d}$.

(2) Let G be an abelian group written additively and $\mathcal{F}(G)$ the free abelian monoid on G. We write the elements of $\mathcal{F}(G)$ in the form $X = g_1 \cdots g_l = \prod_{g\in G} g^{v_g(X)}$. The submonoid

$$\mathcal{B}(G) = \left\{\prod_{g\in G} g^{v_g} \in \mathcal{F}(G) \mid \sum_{g\in G} v_g g = 0\right\}$$

is known as the *block monoid on* G and its elements are referred to as *blocks* over G. If S is a subset of G, then the submonoid

$$\mathcal{B}(G, S) = \left\{\prod_{g\in G} g^{v_g} \in \mathcal{B}(G) \mid v_g = 0 \text{ if } g \notin S\right\}$$

of $\mathcal{B}(G)$ is called *the restriction of* $\mathcal{B}(G)$ *to* S. If S generates G as a monoid, then $\mathcal{B}(G, S)$ is a Krull monoid with divisor class group G. If $n \geq 2$, then set $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$. Consider the block monoid $\mathcal{B}(\mathbb{Z}_n, \{\overline{1}, \overline{n-1}\})$. It is easy to argue that the irreducible elements of $\mathcal{B}(\mathbb{Z}_n, \{\overline{1}, \overline{n-1}\})$ are

$$\overline{1}^{\,n}; \overline{n-1}^{\,n}; \overline{1}\;\overline{n-1},$$

and the only relation amongst the irreducibles is $\overline{n-1}^{\,n} \cdot \overline{1}^{\,n} = (\overline{1}\;\overline{n-1})^n$. By [21, Lemma 2.8], $\Delta(\mathcal{B}(\mathbb{Z}_n, \{\overline{1}, \overline{n-1}\}) = \{n-2\}$, and hence $\mathrm{LD}(\mathcal{B}(\mathbb{Z}_n, \{\overline{1}, \overline{n-1}\}) = \frac{1}{n-2}$.

(3) The bounds in Proposition 2.3 may be strict in general. Consider the numerical monoid $M = \langle 6, 9, 20\rangle$. Here $\Delta(M) = \{1, 2, 3, 4\}$. Using techniques from [12, 19], it can be shown that $\mathrm{LD}(M) = \mathrm{LD}(60) = \frac{4}{7}$, and that $\mathrm{LD}(x) \to 1$ as $x \to \infty$.

We now work toward the other extreme and start with a definition.

Definition 2.6. We say an atomic monoid M has the *Kainrath Property* if for every nonempty finite subset $L \subset \{2, 3, 4, \ldots\} = \mathbb{N} - \{1\}$ there exists an element $x \in M^{LI}$ such that $\mathsf{L}(x) = L$.

Clearly a monoid M with the Kainrath property satisfies $\Delta(M) = \mathbb{N}$. We also immediately deduce the following.

Corollary 2.7. *If M has the Kainrath property, then $\{\mathrm{LD}(x) \mid x \in M^{LI}\} = (0, 1]$. Hence, $\mathrm{LD}(M) = 0$, and M does not have accepted length density.*

Example 2.8. We now examine several families of Kainrath monoids.

(1) If M is a Krull monoid with infinite divisor class group and a prime divisor in every divisor class, then M satisfies the Kainrath property by [41] and thus $\mathrm{LD}(M) = 0$. The assumption that every divisor class contain a prime divisor is crucial here. For instance, there are Krull domains with divisor class group $\mathbb{Z}$ which are half-factorial domains (see [5]). Recall that if D is a Krull domain with class group G, then $D[X]$ is another Krull domain with class group G, but for $D[X]$ each ideal class of G contains

a prime divisor (see [24, Theorem 14.3]). Thus, if D is a Krull domain with infinite divisor class group, then $D[X]$ has the Kainrath property and hence $\mathrm{LD}(D[X]) = 0$ (even if these two facts do not hold for D).

(2) An additive submonoid P of $\mathbb{Q}_{\geq 0}$ (the nonnegative rationals) is known as a *Puiseux monoid*. These monoids are in some sense a natural generalization of numerical monoids. If S is a numerical monoid, then $|\Delta(S)| < \infty$ by [14, Corollary 2.3]. This fails in general for Puiseux monoids; in fact, by [39, Theorem 3.6], there exists a Puiseux monoid P which has the Kainrath property. Hence $\mathrm{LD}(P) = 0$.

(3) In a recent paper Frisch [25] showed $\mathrm{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] \mid f(z) \in \mathbb{Z} \text{ for all } z \in \mathbb{Z}\}$, called the ring of integer-valued polynomials over $\mathbb{Z}$, is Kainrath. So, $\mathrm{LD}(\mathrm{Int}(\mathbb{Z})) = 0$.

(4) By [23, Proposition 4.9 and Theorem 4.10] for every $n \in \mathbb{N}$, the power monoid $\mathcal{P}_{fin,0}(\mathbb{N})$ has an element x_n with $\Delta(x_n) = \{n\}$; hence $\mathrm{LD}(x_n) = \frac{1}{n}$. Consequently, we obtain $\mathrm{LD}(\mathcal{P}_{fin,0}(\mathbb{N})) = 0$ but it is not accepted. It is open whether or not $\mathcal{P}_{fin,0}(\mathbb{N})$ has the Kainrath property (see [23, page 292]).

We close this section by constructing two extremal monoids: one with rational length density that is not accepted (Example 2.11), and another with infinite delta set but positive length density (Example 2.12).

Lemma 2.9. *If $x, y \in M^{LI}$ with $L(xy) = L(x) + L(y)$ and $l(xy) = l(x) + l(y)$, then*

$$\mathrm{LD}(xy) \geq \min(\mathrm{LD}(x), \mathrm{LD}(y)).$$

Moreover, this inequality is strict if $\mathrm{LD}(x) \neq \mathrm{LD}(y)$.

Proof. We have $\mathsf{L}(xy) \supseteq \mathsf{L}(x) + \mathsf{L}(y)$, since we can always factor xy by factoring x and y separately, and concatenating. So, $|\mathsf{L}(xy)| \geq |\mathsf{L}(x)| + |\mathsf{L}(y)| - 1$, by a simple observation on sizes of set sums ($|A + B| \geq |A| + |B| - 1$). Hence

$$\mathrm{LD}(xy) = \frac{|\mathsf{L}(xy)| - 1}{L(xy) - l(xy)} \geq \frac{(|\mathsf{L}(x)| - 1) + (|\mathsf{L}(y)| - 1)}{(L(x) - l(x)) + (L(y) - l(y))} = \mathrm{LD}(x) \oplus \mathrm{LD}(y)$$

where $\oplus$ denotes the mediant $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. A well-known property of the mediant is that $\frac{a}{b} \oplus \frac{c}{d}$ lies in the interval between $\frac{a}{b}$ and $\frac{c}{d}$ (in the interior unless $\frac{a}{b} = \frac{c}{d}$). □

Theorem 2.10. *For any collection of monoids M_i, we have*

$$\mathrm{LD}\big(\textstyle\bigoplus_i M_i\big) = \inf_i(\mathrm{LD}(M_i)).$$

Proof. For any formal product $\prod_i x_i \in \bigoplus_i M_i$ of elements with each $x_i \in M_i$ and at least one $x_i \in M_i^{LI}$, the hypotheses of Lemma 2.9 are satisfied, implying $\mathrm{LD}\big(\bigoplus_i M_i\big) \geq \inf_i(\mathrm{LD}(M_i))$. Conversely, the image of any $x \in M_i$ in $\bigoplus_i M_i$ has identical length set, and thus equal length density. This completes the proof. □

Example 2.11. Fix $i \geq 3$, and let M_i be the quotient of the free abelian monoid on atoms $a_1, \ldots, a_i$ by the relations

$$a_1^3 = a_2^4 = a_3^6 = a_4^8 = \cdots = a_i^{2i}.$$

In other words, M_i is the monoid presented by the i atoms $a_1, \ldots, a_i$ as generators with the relations listed above. Letting $M = \bigoplus_{i \geq 3} M_i$, it is clear that $\Delta(x) = \{1, 2\}$ for every $x \in M$ with nonunique factorization, meaning the length density $\mathrm{LD}(M) = \frac{1}{2}$ obtained from Theorem 2.10 is not accepted.

Example 2.12. For each $i \geq 2$, it is not hard to show, again using techniques from [12, 19], that the numerical monoid $M_i = \langle 2i, 3i, 6i+1 \rangle$ has $\mathrm{LD}(M_i) = \frac{1}{2}$ achieved at $x_i = i(6i+1) \in M_i$, which has length set

$$\mathsf{L}_{M_i}(x_i) = \{i\} \cup \{2i+1, 2i+2, \ldots, 3i\}.$$

As such, $M = \bigoplus_{i \geq 2} M_i$ satisfies $|\Delta(M)| = \infty$ but $\mathrm{LD}(M) = \frac{1}{2} > 0$.

3. Nonrational and Accepted Length Density

A fundamental question early in the study of elasticity was whether or not an integral domain can have irrational elasticity. In [2, Theorem 3.2] the authors show that for any real number $\alpha > 1$, there is a Dedekind domain D with $\rho(D) = \alpha$ (we note that if $\alpha \notin \mathbb{Q}$, then D must necessarily have infinite class group). We now prove a somewhat similar result for length density, but use a completely different construction in the spirit of Example 2.11.

Let $a, b \in \mathbb{N}$ with $b > a$. Let $c \in [0, 1]$. For each $i \in \mathbb{N}$, set $k(i) = \lceil ic(b-a) \rceil$. We will now define the monoid $M(a, b, c)$, as the quotient of the free abelian monoid on atoms $\{q_{i,j} : i, j \in \mathbb{N}\}$, with minimal relations:

$$\forall i \in \mathbb{N},\ q_{i,ia}^{ia} = q_{i,ia+1}^{ia+1} = q_{i,ia+2}^{ia+2} = \cdots = q_{i,ia+k(i)}^{ia+k(i)} = q_{i,ib}^{ib}.$$

Proposition 3.1. *If $a < b \in \mathbb{N}$ and $c \in [0, 1]$, then $\rho(M(a, b, c)) = \frac{b}{a}$ and $\mathrm{LD}(M(a, b, c)) = c$.*

Proof. Set $M = M(a, b, c)$. We first observe that for any $t \in \mathbb{N}$, we have

$$tic(b-a) + 1 \leq tk(i) + 1 < tic(b-a) + 1 + t,$$

and hence

$$c + \frac{1}{ti(b-a)} \leq \frac{tk(i)+1}{ti(b-a)} < c + \frac{t+1}{ti(b-a)}. \tag{6}$$

Note that each atom appears in at most one minimal relation. We may thus calculate that $\mathsf{L}(q_{i,ia}^{ia}) = \{ia, ia+1, ia+2, \ldots, ia+k(i), ib\}$. We have $\rho(q_{i,ia}^{ia}) = \frac{ib}{ia} = \frac{b}{a}$. This proves that $\rho(M) \geq \frac{b}{a}$. We also have $\mathrm{LD}(q_{i,ia}^{ia}) = (k(i)+1)/(ib-ia)$. By (6) with $t = 1$, $\mathrm{LD}(q_{i,ia}^{ia}) > c$ and also $\mathrm{LD}(q_{i,ia}^{ia}) \to c$ as $i \to \infty$. This proves that $\mathrm{LD}(M) \leq c$.

Now, take $t \in \mathbb{N}$, and consider $\mathsf{L}(q_{i,ia}^{tia})$. The minimum element is tia and the maximum is tib. It contains the interval $[tia, tia + tk(i)]$, being the set sum of t intervals.

Consider now an arbitrary element $x \in M$. We write $x = x' \prod_{i \in \mathbb{N}} q_{i,ia}^{x(i)ia}$, where we choose $x(i) \in \mathbb{N}_0$ to be maximal. The leftover atoms, those dividing x', are all inert, as x' is the gcd of all factorizations of x. We have

$$\mathsf{L}(x) = \{|x'|\} + \sum_{i \in \mathbb{N}} \mathsf{L}(q_{i,ia}^{x(i)ia}) \supseteq \{|x'|\} + \sum_{i \in \mathbb{N}} [x(i)ia,\ x(i)(ia + k(i))].$$

Now, the set sum of intervals is also an interval, so $\mathsf{L}(x)$ contains the interval

$$I = \left[|x'| + \sum_{i \in \mathbb{N}} x(i)ia,\ |x'| + \sum_{i \in \mathbb{N}} x(i)(ia + k(i)) \right].$$

Also, $\mathsf{L}(x)$ contains $|x'|+\sum_{i\in\mathbb{N}} ibx(i)$ as a maximal element, and $|x'|+\sum_{i\in\mathbb{N}} iax(i)$ as a minimal element, so we calculate

$$\rho(x) = \frac{|x'| + \sum_{i\in\mathbb{N}} ibx(i)}{|x'| + \sum_{i\in\mathbb{N}} iax(i)} \le \frac{\sum_{i\in\mathbb{N}} ibx(i)}{\sum_{i\in\mathbb{N}} iax(i)} = \frac{b}{a}.$$

So, since $\mathsf{L}(x)$ contains the interval I, of length $|I| = 1 + \sum_{i\in\mathbb{N}} x(i)k(i)$,

$$\mathrm{LD}(x) \ge \frac{\sum_{i\in\mathbb{N}} x(i)k(i)}{\left(|x'| + \sum_{i\in\mathbb{N}} ibx(i)\right) - \left(|x'| + \sum_{i\in\mathbb{N}} iax(i)\right)} \ge \frac{\sum_{i\in\mathbb{N}} x(i)ic(b-a)}{(b-a)\sum_{i\in\mathbb{N}} ix(i)} = c.$$

Since $x \in M$ was arbitrary, this proves that $\rho(M) \le \frac{b}{a}$ and $\mathrm{LD}(M) \ge c$. This establishes the desired result when combined with the opposite inequalities, established previously. □

We begin to explore the question of when a monoid M has accepted length density. We first consider monoids with accepted length density equal to the left hand side of inequality (5). Before proceeding, we will need some definitions. Suppose that M is a commutative cancellative atomic BF-monoid. Without loss for purposes of considering length sets, we assume that M is reduced (i.e., has a unique unit). Let $\mathcal{Z}(M)$ be the free abelian monoid on the atoms of M, which is called the *factorization monoid of* M. There is a natural map $\pi : \mathcal{Z}(M) \to M$ that sends a factorization on to its relevant element in M. If $m \in M$, then set $\mathsf{Z}_M(m) = \pi^{-1}(m)$ which is known as the set of factorizations of m in M. Two factorizations $z, y \in \mathsf{Z}_M(m)$ can be written in terms of atoms as $z = u_1 \cdots u_k v_1 \cdots v_\ell$ and $y = u_1 \cdots u_k w_1 \cdots w_n$, where $\{v_1, \ldots, v_\ell\} \cap \{w_1, \ldots, w_n\} = \emptyset$. Set $\gcd(z, y) = u_1 \cdots u_k \in \mathcal{Z}(M)$, and the distance between z and y to be $\mathbf{d}(z, y) = \max\{n, \ell\} \in \mathbb{N}_0$. For each nonunit $m \in M$, the *factorization graph* ∇_m has vertex set $\mathsf{Z}_M(m)$, and two vertices $z, y \in \mathsf{Z}_M(m)$ share an edge if $\gcd(z, y) \ne 0$. If ∇_m is not connected, then m is called a Betti element of M. Write

$$\mathrm{Betti}(M) = \{b \in M^{LI} \mid \text{ if } \nabla_b \text{ is disconnected}\}$$

for the set of Betti elements of M. It is known that any two distinct factorizations $x, y \in \mathsf{Z}(m)$ are connected by a sequence of trades, each of which occurs between factorizations in different connected components of ∇_b for some Betti element $b \in \mathrm{Betti}(M)$. More precisely, there exist Betti elements $b_1, \ldots, b_k$ and factorizations $z_0, \ldots, z_k \in \mathsf{Z}(m)$ with $z_0 = x$ and $z_k = y$ such that for each i, $z_{i-1} - \gcd(z_{i-1}, z_i)$ and $z_i - \gcd(z_{i-1}, z_i)$ are factorizations in different connected components of ∇_{b_i} (see [13]).

Proposition 3.2. *If M has accepted length density and $|\Delta(M)| < \infty$, then $\mathrm{LD}(M) = \frac{1}{\max\Delta(M)}$ if and only if $\mathrm{LD}(b) = \frac{1}{\max\Delta(M)}$ for some $b \in \mathrm{Betti}(M)$.*

Proof. By the hypothesis, $|\Delta(M)| < \infty$, so let $\delta = \max\Delta(M)$. The backward implication follows immediately from the bound $\mathrm{LD}(M) \ge 1/\delta$. Conversely, suppose $\mathrm{LD}(M) = 1/\delta$. By assumption, there exists some $m \in M$ such that $\mathrm{LD}(m) = 1/\delta$. Since $|\mathsf{L}(m)| \ge 2$, there exists $b \in \mathrm{Betti}(M)$ dividing m in M with $|\mathsf{L}(b)| \ge 2$, meaning $\mathsf{L}(b) + c \subseteq \mathsf{L}(m)$ for some $c \in \mathbb{N}_0$. In particular, $\Delta(m)$ must have some element at most $\max\Delta(b)$. Since $\delta = \max\Delta(M)$, we have $\max\Delta(M) \le \delta$, so $\mathrm{LD}(m) = \frac{1}{\delta} \le \frac{1}{\max\Delta(m)} \le \mathrm{LD}(M)$ by equation (4). Hence, $\delta = \max\Delta(M)$ and since we have equality in equation (4), Proposition 2.3 forces $|\Delta(m)| = 1$, so $\Delta(m) = \{\delta\}$. We conclude as well that $\Delta(b) = \{\delta\}$ and $\mathrm{LD}(b) = 1/\delta$. □

Example 3.3. Length density need not be attained at a Betti element. Indeed, the numerical semigroup $M = \langle 20, 28, 42, 73\rangle$ has $\mathrm{Betti}(M) = \{84, 140, 146\}$ with length sets $\{2, 3\}$, $\{4, 5, 7\}$, and $\{2, 4, 5\}$, respectively, but $\mathsf{L}(202) = \{4, 6, 7, 9\}$ yields the length density $\frac{3}{5} < \frac{2}{3}$.

Under various conditions on the monoid M, there are established structure theorems for the set $\mathsf{L}(x)$ where x is a nonunit of M. Let $L \subset \mathbb{Z}$ be finite, $d \in \mathbb{N}$, and $l, M \in \mathbb{N}_0$. We call L an *almost arithmetic progression* (or *AAP*) with difference d, length l, and bound M if

$$L = y + (L' \cup L^* \cup L'') \subseteq y + d\mathbb{Z}$$

where $L^* = d\mathbb{Z} \cap [0, ld]$, $L' \subseteq [-M, -1]$, $L'' \subseteq ld + [1, M]$, and $y \in \mathbb{Z}$. An analysis of monoids and elements whose length sets are almost arithmetic progressions can be found in [33, Chapter 4].

Theorem 3.4. *If M is finitely generated, then* $\mathrm{LD}(M)$ *is accepted.*

Proof. Suppose $m_1, m_2, \ldots \in M$ satisfy $\mathrm{LD}(m_i) > \mathrm{LD}(m_{i+1})$ for every i. To complete the proof, it suffices to locate an element $a \in M$ such that $\mathrm{LD}(a) < \mathrm{LD}(m_i)$ for all i. In particular, this proves that every decreasing sequence of length densities has a lower bound, and thus that the set of length densities must contain its infimum.

Since M is fnitely generated, the structure theorem for sets of length [29, Proposition 8.1] implies there exists an integer d, dependent only on M and divisible by every element of $\Delta(M)$, such that for any nonunit $m \in M$, we have

$$\mathsf{L}(m) = \big((D + d\mathbb{Z}) \cap [\ell, L]\big) \setminus \big((D' + \ell) \cup (D'' + L - d)\big)$$

for some sets $D, D', D'' \subset [1, d]$ and integers $\ell, L \geq 1$. Since there are only finitely many possible sets D, D', and D'', by choosing an appropriate subsequence of the m_i, it suffices to assume the length set of each m_i can be expressed above with the same D, D', and D'', and therefore only differ in the values of ℓ and L. With this, it is not hard to see that

$$\lim_{i\to\infty} \mathrm{LD}(m_i) = \frac{|D|}{d}.$$

Since $\mathrm{LD}(m_i)$ is a strictly decreasing sequence and $\sup \Delta(M) < \infty$, there exists a gap size δ satisfying (i) the number of times δ occurs as a gap size between sequential lengths in $\mathsf{L}(m_i)$ is an unbounded sequence, and (ii) $\mathrm{LD}(m_i) > 1/\delta$ for all i. Moreover, since M is finitely generated, by [42, Theorem 4.9] there are only finitely many $b \in M$ having factorizations $x, y \in \mathsf{Z}(b)$ with disjoint support, sequential lengths $|x|$, $|y| \in \mathsf{L}(b)$, and length difference $|x| - |y| = \delta$. As such, by again choosing an appropriate subsequence of the m_i, we can assume that there exists an element $b \in M$, independent of i, such that for each i, the trade $x \sim y$ occurs at least i times between sequential length factorizations of m_i (that is, there exist at least i pairs of factorizations $x', y' \in \mathsf{Z}(m_i)$ with sequential lengths in $\mathsf{L}(m_i)$ such that $x' - y' = x - y$).

Now, by the above setup, we have $b \mid m_i$ for each i, so write each $m_i = ba_i$ for $a_i \in M$. By the above reasoning, choosing an appropriate subsequence of the a_i (and thus of the m_i), it suffices to assume there exist sets E, E', and E'', independent of i, such that for each i,

$$\mathsf{L}(a_i) = \big((E + d\mathbb{Z}) \cap [\ell, L]\big) \setminus \big((E' + \ell) \cup (E'' + L - d)\big)$$

for some integers $\ell, L \geq 1$. Since $|x| + \mathsf{L}(a_i) \subseteq \mathsf{L}(m_i)$ for all i, we must have $|E| \leq |D|$. There are now two cases to consider. First, if D is an arithmetic sequence of step size δ, then we are done since $\Delta(b) = \{\delta\}$ and thus $\mathrm{LD}(b) = 1/\delta < \mathrm{LD}(m_i)$ for every i. Otherwise, at some point in D (and thus an arbitrarily large number of times in $\mathsf{L}(m_i)$ as $i \to \infty$), a length gap of δ is immediately preceded by a length gap strictly smaller than δ. This implies $|E| < |D|$ and thus

$$\lim_{i\to\infty} \mathrm{LD}(a_i) = \frac{|E|}{d} < \frac{|D|}{d} = \lim_{i\to\infty} \mathrm{LD}(m_i),$$

thereby completing the proof. □

Example 3.5. In any finitely generated semigroup, the structure theorem for sets of length implies the set of elasticities must have both a maximum and a minimum, and that any infinite strictly monotone sequence of elasticities must be increasing. The same need not hold for length density, as the following examples illustrate.

First, consider the numerical semigroup $M = \langle 6, 10, 15\rangle$, wherein one can check that

$$\mathsf{L}_M(30n) = ([2n, 5n] \cap \mathbb{Z}) \setminus \{5n - 1\},$$

which yields a strictly increasing sequence of length densities. Moreover, one can use the fact that M has a unique Betti element [28] to argue that no element of M has length density $1 = 1/\min \Delta(M)$, meaning the set of length densities of M has no maximum.

On the other hand, consider the numerical semigroup $S = \langle 4, 7\rangle$, so

$$\mathsf{L}_S(28n) = \{4n + 3k \mid 0 \le k \le n\}.$$

Then, take $T = \langle(4, 0, 0), (7, 0, 0), (0, 3, 0), (0, 1, 1), (0, 0, 3)\rangle \subset \mathbb{N}_0^3$, so

$$\mathsf{L}_T((28n, 3, 3)) = \mathsf{L}_S(28n) + \{2, 3\}$$

has gap sequence $1, 2, 1, 2, \ldots, 1, 2, 1$, producing a length density sequence

$$\mathrm{LD}_T((28n, 3, 3)) = (2n + 1)/(3n + 1)$$

that is strictly decreasing. Despite this, the length density of T is accepted by Theorem 3.4, and in particular $\mathrm{LD}(T) = \mathrm{LD}_T((28, 0, 0)) = \frac{1}{3}$.

Example 3.6. We consider some examples related to Theorem 3.4.

(1) If G is a finite abelian group and S a nonempty subset of G, then the block monoid $\mathcal{B}(G, S)$ is finitely generated and hence has accepted length density. For a monoid M, set $\mathcal{L}(M) = \{\mathsf{L}(x) \mid x \in M^{LI}\}$. If M is a Krull monoid with divisor class group G and distribution of primes S, then by [31, Proposition 12] we have that $\mathcal{L}(M) = \mathcal{L}(\mathcal{B}(G, S))$. Hence, if G is finite abelian, then M has accepted length density. This class includes all Krull domains with finite divisor class group, which includes the ring of algebraic integers in a finite extension of the rationals.

(2) Since numerical monoids are finitely generated, they also have accepted length density. While the computation of the elasticity of a numerical monoid M is relatively simple [17, Theorem 2.1], the computation of its length density is a more complex calcuation and we defer for the time being an extended study of this question.

(3) An atomic integral domain D is a *Cohen-Kaplansky domain* (or a *CK-domain*) if it has finitely many nonassociated irreducible elements. In [4, Theorem 4.3], the authors give 14 conditions equivalent to D being a CK-domain; the most notatable among these being D is a one-dimensional semilocal domain such that for each nonprincipal maximal ideal M of D, D/M is finite and D is analytically irreducible. Theorem 3.4 implies that a CK-domain has nonzero accepted length density.

We briefly approach the question of computing $\mathrm{LD}(\mathcal{B}(G))$ and start with a known result concerning the delta set of such a block monoid. If $G = \oplus_{i=1}^{k} \mathbb{Z}_{n_i}$ is a finite abelian group where $n_i | n_{i+1}$ for $1 \leq i < k$ with $|G| \geq 3$, then by [30, Corollary 2.3.5]

$$[1, n_k - 2] \subseteq \Delta(\mathcal{B}(G)) \subseteq [1, \mathsf{c}(\mathcal{B}(G)) - 2] \subseteq [1, \mathsf{D}(G) - 2]. \tag{7}$$

Here $\mathsf{D}(G)$ represents the *Davenport Constant* of G; this is the longest length of a nonzero sequence which sums to 0, but has no proper subsum that sums to zero. The quantity $\mathsf{c}(M)$ is the *catenary degree* of the monoid M, which we define generally as follows. If $a \in M$ and $z_0, z_1, \ldots, z_k \in \mathsf{Z}_M(a)$, then $z_0, z_1, \ldots, z_k$ is called a *chain from* z_0 to z_k. For $N \in \mathbb{N}$, $z_0, z_1, \ldots, z_k$ is called an N-chain if $\mathbf{d}(z_i, z_{i+1}) \leq N$ for $0 \leq i \leq k-1$. The catenary degree, denoted $\mathsf{c}(a)$ of $a \in M$ is the smallest $N \in \mathbb{N}_0 \cup \infty$ such that any two factorizations z, y in $\mathsf{Z}_M(a)$ can be linked by an N-chain. We then set $\mathsf{c}(M) = \sup\{\mathsf{c}(a) \mid a \in M\}$. Equation (7) immediately leads to the following.

Proposition 3.7. *If G is a finite abelian group with $|G| \geq 3$, then*

$$\frac{1}{\mathsf{c}(\mathcal{B}(G)) - 2} \leq \mathrm{LD}(\mathcal{B}(G)) \leq 1.$$

It is known for each finite abelian group G that $\Delta(\mathcal{B}(G))$ is a complete interval (i.e., $\Delta(\mathcal{B}(G)) = \{1, 2, \ldots, \max \Delta(\mathcal{B}(G))\}$) (see [36]). The containments

$$\Delta(\mathcal{B}(G)) \subseteq [1, \mathsf{c}(\mathcal{B}(G)) - 2] \subseteq [1, \mathsf{D}(G) - 2]$$

form an equality if and only if G is cyclic or an elementary 2-group [37, Theorem A]. Thus, in either of these two cases we have $\Delta(\mathcal{B}(G)) = \{1, 2, \ldots, \mathsf{D}(G) - 2\}$. For a cyclic group, this yields that $\Delta(\mathcal{B}(\mathbb{Z}_n)) = \{1, 2, \ldots, n-2\}$, and for an elementary 2-group it yields that $\Delta(\mathcal{B}(\oplus_{i=1}^{k} \mathbb{Z}_2)) = \{1, 2, \ldots, k-1\}$ (using the known formula for $\mathsf{D}(\oplus_{i=1}^{k} \mathbb{Z}_2) = k+1$). Combining this with Proposition 2.3, we obtain that

$$\frac{1}{n-2} \leq \mathrm{LD}(\mathcal{B}(\mathbb{Z}_n)) \leq 1 \text{ and } \frac{1}{k-1} \leq \mathrm{LD}(\mathcal{B}(\oplus_{i=1}^{k} \mathbb{Z}_2)) \leq 1.$$

By [30, Corollary 2.3.6], if G is either cylic or an elementary 2-group, then some block $B \in \mathcal{B}(G)$ has $\mathsf{L}(B) = \{2, \mathsf{D}(G)\}$. Hence, in both cases $\mathrm{LD}(B) = \frac{1}{\mathsf{D}(G)-2}$ which yields the following.

Proposition 3.8. *If $G = \mathbb{Z}_n$ is cyclic, then $\mathrm{LD}(\mathcal{B}(\mathbb{Z}_n)) = \frac{1}{n-2}$, and if $G = \oplus_{i=1}^{k} \mathbb{Z}_2$, then $\mathrm{LD}(\mathcal{B}(\oplus_{i=1}^{k} \mathbb{Z}_2)) = \frac{1}{k-1}$.*

We list an application of Proposition 3.8 to algebraic rings of integers.

Corollary 3.9. *Any ring R of algebraic integers with prime class number p has $\mathrm{LD}(R) = \frac{1}{p-2}$.*

In closing this section, we note that progress on the computation of further values of $\mathrm{LD}(\mathcal{B}(G))$ depends on improved computations of $\mathsf{c}(\mathcal{B}(G))$. Outside of the groups listed in Proposition 3.8, $\mathrm{c}(\mathcal{B}(G))$ is known for only a handful of groups (see [37, Theorems A and 1.1]). Hence, we leave an extended discussion of this problem to future consideration.

4. Asymptotic Length Density

We open this section by constructing an atomic monoid with an element that lacks asymptotic length density.

Example 4.1. Consider the Puiseux monoid

$$M = \left\langle \frac{4}{3}, \frac{8}{5}, \frac{800}{1201}, \frac{a_1}{p_1}, \frac{a_2}{p_2}, \dots \right\rangle.$$

The p_i are a strictly increasing sequence of primes, to be specified later. The a_i are a strictly increasing sequence of natural numbers, to be specified later. Because the a_i are distinct, all of the atoms of M are unstable, so by [40, Theorem 4.8], M is a BF-monoid (and an FF-monoid).

Our focus is on $x = 8$, and we calculate $x^n = 8n$, as n grows large. For $n < 100$, $x^n < 800$. Hence, all factorizations of x^n will include only the first two atoms. Note that $8 = 6 \cdot \frac{4}{3} = 5 \cdot \frac{8}{5}$, so $\mathsf{L}(x^n) = \{5, 6\}^n = [5n, 6n]$. In particular $\mathrm{LD}(x^n) = 1$.

At $n = 100$, $x^n = 800$, and we get the new factorization $800 = 1201 \cdot \frac{800}{1201}$ and hence $800 = 600 \cdot \frac{4}{3} = 500 \cdot \frac{8}{5} = 1201 \cdot \frac{800}{1201}$. Note that 1201 was chosen to be the smallest prime greater than twice 600. Hence we have $\mathsf{L}(x^{100}) = [500, 600] \cup \{1201\}$ so $\mathrm{LD}(x^{100}) < \frac{1}{2}$.

As n continues to increase, so long as $8n < a_1$, all factorizations of x^n will include only the first three atoms. Note that we have the trade $600 \cdot \frac{4}{3} = 500 \cdot \frac{8}{5} = 1201 \cdot \frac{800}{1201}$. We take $n = 100k$ for an integer $k \geq 2$ and calculate the length set of x^{100k} as the union of intervals. We have

$$\mathsf{L}(x^{100k}) = [500k, 600k] \cup [500k + 701, 600k + 601] \cup [500k + 2 \cdot 701, 600k + 2 \cdot 601] \cup \cdots.$$

The last interval will be $[500k + k \cdot 701, 600k + k \cdot 601] = \{1201k\}$. Note that if $600k \geq (500k + 701)$, i.e. $100k \geq 701$, then the first two intervals overlap. If $600k + 601 > 500k + 2 \cdot 701$, i.e $100k \geq 801$, then the first three intervals overlap. If $600k + i \cdot 601 > 500k + (i+1) \cdot 701$, i.e. $100k > 701 + 100i$, then the i-th interval overlaps with the $(i+1)$-th interval. Taking $i = \frac{3}{4}k$, if $k \geq 29 > \frac{701}{25}$, we have the i-th interval overlapping with the $(i+1)$-th interval. In particular, $\mathsf{L}(x)$ will contain the interval $[500k, 600k + \frac{3}{4}k \cdot 601]$, i.e. $[500k, 1050.75k]$. Hence $\mathrm{LD}(x^{2900}) \geq \frac{550.75 \cdot 29 - 1}{701 \cdot 29} > \frac{3}{4}$.

We are now ready to choose the next atom. Set $a_1 = 2901 \cdot 8$, and $p_1 > 2 \cdot 1201 \cdot 30$, e.g. $p_1 = 72073$. Using only the first three atoms, all factorizations of x^{2901} are of length at most $30 \cdot 1201$. Using the new fourth atom, we get a new factorization of length p_1, which gives a gap of length at least $30 \cdot 1201$. Hence $\mathrm{LD}(x^{2901}) < \frac{1}{2}$.

Continuing in this way, we find $\mathrm{LD}(x^n)$ can be made to grow to be above $\frac{3}{4}$, then to shrink below $\frac{1}{2}$, over and over as $n \to \infty$. Hence the asymptotic length density of x does not exist.

What atomic monoids admit asymptotic length densities for all their elements? While we do not completely answer this, we offer a large class that does. We again require some definitions. If $L \subset \mathbb{N}_0$, then set

$$\rho(L) = \sup\left\{ \frac{m}{n} \mid m, n \in L \right\} = \frac{\sup L}{\min L} \in \mathbb{Q}_{\geq 1} \cup \{\infty\}.$$

If H is a BF-monoid and $\Delta(H) \neq \emptyset$, then $d = \min \Delta(H) = \gcd(\Delta(H))$ by [33, Proposition 1.4.4]. Hence, $\Delta(H) \subseteq d\mathbb{N}$, and if $\Delta(H)$ is finite, then $\rho(\Delta(H)) \in \mathbb{N}$.

If M is a monoid and $x \in M$, then let $[\![x]\!]$ denote the set of all elements in M that divide x^k for some $k \in \mathbb{N}$. For $a \in M$ and $x \in \mathcal{Z}(M)$, let $\mathsf{t}(a, x) \in \mathbb{N}_0 - \{1\}$ denote the smallest $N \in \mathbb{N}_0 - \{1\}$ with the following property: If $\mathsf{Z}_M(a) \cap x\mathcal{Z}(M) \neq \emptyset$ and $z \in \mathsf{Z}_M(a)$, then there exists some factorization $z' \in \mathsf{Z}_M(a) \cap x\mathcal{Z}(M)$ such that $\mathbf{d}(z, z') \le N$. If $\mathsf{Z}_M(a) \cap x\mathcal{Z}(M) = \emptyset$, then $\mathsf{t}(a, x) = 0$. We call $\mathsf{t}(a, x)$ the *tame degree* of a with respect to x. For any subsets $M' \subset M$ and $X \subset \mathsf{Z}(M)$, we define

$$\mathsf{t}(M', X) = \sup\{\mathsf{t}(a, x) \mid a \in M', x \in X\} \in \mathbb{N}_0 \cup \{\infty\}.$$

In particular, for $a \in M$ we define $\mathsf{t}(a, X) = \mathsf{t}(\{a\}, X)$, and for $x \in X$ we define $\mathsf{t}(M', x) = \mathsf{t}(M', \{x\})$. The monoid M is *locally tame* if $\mathsf{t}(M, u) < \infty$ for each atom u of M. By [33, Theorem 1.6.7], if M is an atomic locally tame monoid, then M is a BF-monoid. The *tame degree* of M, is defined by $\mathsf{t}(M) = \sup\{\mathsf{t}(M, u) \mid u \in \mathcal{A}(M)\}$. If $\mathsf{t}(M) < \infty$, then M is called *globally tame*. Since $\mathsf{c}(H) \le \mathsf{t}(H)$ by [33, Theorem 1.6.6], global tameness implies finiteness of the catenary degree.

Theorem 4.2. *Let S be a locally tame atomic monoid and for $x \in S$ set $H = [\![x]\!]$. Assume for $x \in S$ that $\Delta(x) \neq \emptyset$ and $|\Delta(H)| < \infty$. Let $d = \min \Delta(H)$, τ be minimal such that $d \in \Delta(x^\tau)$, $\psi = \max(\tau, \rho(\Delta(H)) - 1)$, and $T = \mathsf{t}(H, \mathsf{Z}_S(x^\psi))$. For all $n \ge \psi$ it follows that $\frac{1}{d} - \frac{2T}{nd^2} \le \mathrm{LD}(x^n) \le \frac{1}{d}$. In particular, $\overline{\mathrm{LD}}(x) = 1/d$.*

Proof. Note that [33, Theorem 1.6.7.2] implies the finiteness of T; $|\Delta(H)| < \infty$ yields that ψ is finite. We first prove that H is locally tame (i.e., $\mathsf{t}(H, u)$ is finite for each $u \in \mathcal{A}(H_{\text{red}})$). We have $\mathcal{A}(H_{\text{red}}) \subseteq \mathcal{A}(S_{\text{red}})$. Since $\mathsf{t}(H, u) = \sup\{\mathsf{t}(a, u) \mid a \in H\}$ and $\mathsf{t}(S, u) = \sup\{\mathsf{t}(a, u) \mid a \in S\}$ is finite, then $\mathsf{t}(H, u)$ is finite for each $u \in \mathcal{A}(H_{\text{red}})$ because H is a subset of S. This follows since H divisor closed in S yields for any $a \in H$ that $\mathsf{Z}_S(a) = \mathsf{Z}_H(a)$ and hence $\mathsf{t}_H(a, u) = \mathsf{t}_S(a, u)$.

Let $y \in H$ with $d \in \Delta(y)$. That is, there are factorizations z', z'' of y with $|z''| = |z'| + d$ and no factorizations of y of intermediate length. Since $y \in H$, there is some $m \in \mathbb{N}$ with $y \mid x^m$. Let z''' be any factorization of $y^{-1}x^m$. Hence, $\mathsf{L}(x^m)$ contains $|z'''z''|$ and $|z'''z'|$, which are d apart. It cannot contain a factorization length in between, else d would not be the minimum of $\Delta(H)$. Hence $\Delta(x^m)$ contains d. This proves that τ exists.

Because (see [33, Definition 4.3.1]) $d \in \Delta(x^\tau)$, also $x^\tau \in \Phi(\{0, d\})$ (as is every higher power of x). Recall that $\psi \ge \rho(\Delta(H)) - 1$. We apply [33, Theorem 4.3.6], to get that every multiple of x^ψ (in H) has its length set an AAP with difference d and bound $\mathsf{t}(H, \mathsf{Z}_S(x^\psi)) = T$.

Now take any $n \ge \psi$. Apart from two intervals each of size at most T, every gap in $\mathsf{L}(x^n)$ is of size exactly d. For every $n \ge \psi$, set $Q_n = \max \mathsf{L}(x^n) - \min \mathsf{L}(x^n)$. We have

$$\frac{Q_n - 2T}{d} \le |\mathsf{L}(x^n)| - 1 \le \frac{Q_n}{d},$$

and so

$$\frac{1}{d} - \frac{2T}{Q_n d} \le \mathrm{LD}(x^n) \le \frac{1}{d}.$$

Note that $Q_n \ge nd$, so

$$\frac{1}{d} - \frac{2T}{nd^2} \le \frac{1}{d} - \frac{2T}{Q_n d}.$$

This completes the proof. □

Note the hypothesis that $|\Delta(H)| < \infty$ can be met by $\Delta(S)$ being finite. This happens if the catenary degree is finite (see [33, Theorem 1.6.3]) or, as noted above, if S is globally tame.

Example 4.3. We offer some examples that illustrate Theorem 4.2.

(1) Finitely generated monoids are globally tame (see [33, Theorem 3.1.4]), hence all nonunit elements admit asymptotic length densities. In particular, consider the numerical semigroup $M = \langle 6, 9, 20\rangle$ and $x = 60$. A brief calculation demonstrates that $\mathsf{L}(nx) = [3n, 10n] \setminus \{3n+1, 3n+2, 3n+3\}$, so $\mathrm{LD}(x^n) = \frac{7n-3}{7n} = 1 - \frac{3}{7n}$. Hence the rate of convergence (to the limit of 1), is exactly $O(1/n)$, proving the bound above is tight.
(2) Let H be a Krull monoid with class group G and let $G_0 \subset G$ denote the set of classes containing prime divisors. If the Davenport constant $\mathsf{D}(G_0) < \infty$ (which holds if G_0 is finite), then H is globally tame by [33, Theorem 3.4.10]. Thus such Krull monoids, such as the ring of algebraic integers in a finite extension of the rationals, satisfy Theorem 4.2. Suppose H has infinite cyclic class group G, say $G = \mathbb{Z}$ and let $G_0 \subset G$ denote the set of classes containing prime divisors. If $G_0 \cap \mathbb{N}$ or $G_0 \cap (-\mathbb{N})$ is finite, then H is locally tame by [32, Theorem 4.2] and has finite catenary degree by [9, Theorem 1.1] (we note that H need not be globally tame).
(3) Every C-monoid (see [33, Definition 2.9.5]) is locally tame and has finite catenary degree by [33, Theorem 3.3.4]. Note that every Mori domain D with nonzero conductor $\mathfrak{f}$, finite class group $\mathcal{C}(\widehat{D})$ and finite residue field $\widehat{D}/\mathfrak{f}$ is a C-domain by [33, Theorem 2.11.9]; orders in algebraic number fields are such Mori domains.

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