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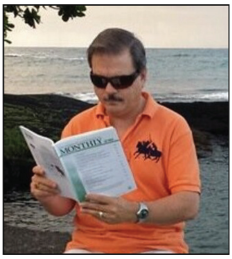
Distances Between Factorizations in the Chicken McNugget Monoid

Scott Chapman, Pedro García-Sánchez, and Christopher O'Neill

Luck is a dividend of sweat. The more you sweat, the luckier you get.
- Ray Kroc [21]



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Prelude

In [15], the authors examined in detail the *Chicken McNugget Monoid* (denoted in that paper by 🐔) and its related factorization properties. These authors preceeded that paper with the following quote from McDonald's founder Ray Kroc [21]: "People just want more of it." From the reaction to that paper, Ray Kroc was right.

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Figure 1. The six-piece box

In the present paper, we will pick up where [15] left off and explore *chains of factorizations* in 🐔. The notion of a chain of factorizations has up to now been largely relegated to research papers and not had wide exposure. Our purpose here is to show, within the context of 🐔, that once the common factorization invariants such as elasticity, sets of length, and delta sets are determined, then a chain of factorizations, which relates to the complete set of factorizations of an element and not just the lengths, is a powerful tool in studying factorization properties. We will introduce a method to measure the distance between two factorizations of a given element (see Definition 2) and from this distance function compute two combinatorial constants: the *catenary degree* (see Definition 6) and the *tame degree* (see Definition 9). While the catenary degree will in some sense measure the total "spread" of the distances between the complete set of factorizations of an element, the tame degree will focus on measuring distances from a factorization to another factorization containing a particular atom. As with [15], we present the definitions and examples in terms of a general numerical monoid, and conclude by specializing our results to the Chicken McNugget Monoid.

Definitions and basic properties of the McNugget Monoid

So what is the Chicken McNugget Monoid? We briefly review some background material which can be found in greater detail in [15]. Chicken McNuggets were originally sold in packages of size 6, 9, or 20 pieces, and the question of how many Chicken McNuggets can be bought without breaking apart a package became a popular recreational mathematics question. More specifically, if n Chicken McNuggets can be purchased using whole packages (where n is a positive integer), then there exist nonnegative integers $x_1, x_2, x_3 \in \mathbb{N}_0$ such that

$$n = 6x_1 + 9x_2 + 20x_3.$$

In this case, n is called a *McNugget number* and (x_1, x_2, x_3) is called a *McNugget expansion* of n. As $(30, 0, 0)$, $(0, 20, 0)$, $(15, 10, 0)$, and $(0, 0, 9)$ are all McNugget expansions of $n = 180$, it is clear that McNugget expansions of a given McNugget number need not be unique. A full list of the McNugget expansions of McNugget numbers up to $n = 50$ can be found in [15, Table 1].

Let

$$\langle 6, 9, 20\rangle = \{6x_1 + 9x_2 + 20x_3 : x_1,\ x_2,\ x_3 \in \mathbb{N}_0\}$$

represent the complete set of McNugget numbers. Under regular integer addition, $\langle 6, 9, 20\rangle$ forms a *monoid*, meaning the sum of any two McNugget numbers is again a McNugget number. As previously advertised, we will call this monoid the *Chicken McNugget monoid* and denote it by 🐔. In more generality, if $n_1, \ldots, n_k$ is a set of relatively prime positive integers, then

$$\langle n_1, \ldots, n_k\rangle = \{x_1n_1 + x_2n_2 + \cdots + x_kn_k\ :\ x_1, \ldots, x_k \in \mathbb{N}_0\}$$

Figure 2. The nine-piece box

is known as a *numerical monoid*. A good general reference on numerical monoids (sometimes called numerical semigroups) is [**25**]. Given $n_1, \ldots, n_k$ as above, it is easy using elementary number theory to argue that there is a largest positive integer not contained in $\langle n_1, \ldots, n_k \rangle$. This positive integer is known as the *Frobenius number* of $\langle n_1, \ldots, n_k \rangle$ and is the focus of much ongoing mathematics research (see [**24**]). Using [**15**, Proposition 1 and Table 1], it follows that the Frobenius number of 🐓 is 43. This is the largest number of Chicken McNuggets that cannot be ordered using whole boxes of sizes 6, 9, or 20.

In keeping with the usual notation used in commutative algebra, we will refer to the elements $n_1, \ldots, n_k$ as the *irreducible elements* or *atoms* of $\langle n_1, \ldots, n_k \rangle$. A representation of an element $n \in \langle n_1, \ldots, n_k \rangle$ as a sum $n = x_1 n_1 + x_2 n_2 + \cdots + x_k n_k$ of atoms will be called a *factorization* of n in $\langle n_1, \ldots, n_k \rangle$. (Note that this is different from the "usual" notion of prime factorization of an integer!) We will use the shorthand form $(x_1, \ldots, x_k)$ to represent a factorization of n in $\langle n_1, \ldots, n_k \rangle$. Set

$$\mathsf{Z}(n) = \{(x_1, \ldots, x_k) : n = x_1 n_1 + x_2 n_2 + \cdots + x_k n_k\}$$

to be the complete set of factorizations of n in $\langle n_1, \ldots, n_k \rangle$. If $z = (x_1, \ldots, x_k) \in \mathsf{Z}(n)$, then the *support* of z is the set

$$\operatorname{supp}(z) = \{i \ : \ 1 \le i \le k \text{ and } x_i \ne 0\}.$$

Given a factorization $z = (x_1, \ldots, x_k) \in \mathsf{Z}(n)$, denote by $|z| = x_1 + \cdots + x_k$ the number of atoms (with repetition) used in z. We call $|z|$ the *length* of z. The set

$$\mathcal{L}(n) = \{ \, |z| \ : \ z \in \mathsf{Z}(n)\}$$

is known as the *set of lengths* of n. In the Chicken McNugget Monoid, each factorization z of a McNugget number $n \in$ 🐓 represents a specific combination of packs to purchase exactly n McNuggets, and its length $|z|$ is simply the number of packs used.

Most of the work in [**15**] centers around studying carefully defined *invariants* that measure the size and structure of length sets of McNugget numbers.

Writing the distinct lengths of a given element $n \in$ 🐓 in order, we obtain $\mathcal{L}(n) = \{j_1, j_2, \ldots, j_t\}$ where $j_i < j_{i+1}$ for $i \in \{1, \ldots, t-1\}$. We further write

$$\ell(n) = j_1 \quad \text{and} \quad \mathrm{L}(n) = j_t$$

for the minimum and maximum factorization lengths of n, respectively. The *elasticity* of n is defined as the ratio

$$\rho(n) = \frac{\mathrm{L}(n)}{\ell(n)} \in \mathbb{Q},$$

and the *elasticity* of 🐓 as

$$\rho(\text{🐓}) = \sup\{\rho(n) \ : \ n \in \text{🐓}\}.$$

Intuitively, elasticity measures how "spread out" a monoid's factorization lengths are. The interested reader can find numerous papers in the recent literature that study problems related to elasticity, in both numerical monoids [**4**, **5**, **13**] and more broadly [**2**].

The *delta set* of a McNugget number n is defined by

$$\Delta(n) = \{j_{i+1} - j_i : 1 \leq i \leq t - 1\},$$

and the *delta set* of 🐔 by

$$\Delta(\text{🐔}) = \bigcup_{n \in \text{🐔}} \Delta(n).$$

Intuitively, the delta set records the “gaps” in (or “missing”) factorization lengths. There is a wealth of recent work concerning the computation of the delta set of a numerical monoid [**5**, **7**, **12**, **14**, **17**]. For numerical monoids with three generators, the computation of the delta set is tightly related to Euclid’s extended greatest common divisor algorithm [**18**, **19**].

We now summarize the main results in [**15**, Corollary 9, Theorem 16], which examine the elasticity and delta set of the Chicken McNugget Monoid.

Proposition 1. *Let* $n \in \text{🐔}$.

1. $\rho(\text{🐔}) = \frac{10}{3}$.
2. *If* $n \geq 92$, *then*

$$\Delta(n) = \begin{cases} \{1\} & \text{if } r = 3, 8, 14, 17, \\ \{1, 2\} & \text{if } r = 2, 5, 10, 11, 16, 19, \\ \{1, 3\} & \text{if } r = 1, 4, 7, 12, 13, 18, \\ \{1, 4\} & \text{if } r = 0, 6, 9, 15, \end{cases}$$

   where $n = 20q + r$ *for* $q, r \in \mathbb{N}_0$ *and* $r < 20$.
3. $\Delta(\text{🐔}) = \{1, 2, 3, 4\}$.

Before describing factorization chains, we note that the numerous calculations we will perform require some type of computing support. The calculations we reference can be performed using the numericalsgps package [**16**] for the computer algebra system GAP. Interested readers are referred to that package for details behind the programming we use; a short tutorial devoted to the Chicken McNugget Monoid can be found at https://numerical-semigroups.github.io/Nuggets/.

Relations, trades, and minimal presentations

We usually think of a numerical monoid $\langle n_1, \ldots, n_k \rangle$ in terms of its atoms $n_1, \ldots, n_k$. Although these determine which integers live in $\langle n_1, \ldots, n_k \rangle$ and which do not, from the point of view of an algebraist, these only tell half of the story. The underlying “algebraic structure” of $\langle n_1, \ldots, n_k \rangle$ also depends on the *relations*, or linear dependencies, between $n_1, \ldots, n_k$.

Let us examine what this means in the context of the Chicken McNugget Monoid. The smallest McNugget number with more than one distinct McNugget expansion is $18 \in \text{🐔}$, which has $\mathsf{Z}(18) = \{(3, 0, 0), (0, 2, 0)\}$ since 18 is a multiple of both 6 and 9. This is precisely what is meant by a *relation*, namely a linear equation relating the atoms 6 and 9. This seemingly small observation has implications for most of the elements of 🐔; in **any** factorization $z = (x_1, x_2, x_3)$ of **any** McNugget number $n \in \text{🐔}$, if $x_1 \geq 3$, then we can freely “trade” 3 copies of 6 for 2 copies of 9 to obtain another factorization of n, namely $(x_1 - 3, x_2 + 2, x_3)$. We use the notation $(3, 0, 0) \sim (0, 2, 0)$ to represent this relation, indicating that in 🐔, 3 times the first atom equals 2 times the second.

It is now natural to ask the following question. Suppose you witness a customer ordering, say, 120 Chicken McNuggets, using 10 packs of 6 and 3 packs of 20. What other ways are there to order that same number of Chicken McNuggets? Well, using the relation $(3, 0, 0) \sim (0, 2, 0)$, we obtain at least 3 more ways, yielding

$$(10, 0, 3),\ (7, 2, 3),\ (4, 4, 3),\ \text{and}\ (1, 6, 3).$$

Surely there must be others, since all of the above factorizations use the same number of 20 packs. To obtain these, we need another relation, one that involves the atom 20. Naturally, we should look for the smallest McNugget number that can be expressed using both packs of 20 and packs of 6 and/or 9. It turns out the magic number is $60 \in$ 🐔, which has

$$\mathsf{Z}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}.$$

We are now presented with a choice: which relation do we want? Certainly it must involve the factorization $(0, 0, 3)$, but which of the 4 factorizations involving 6- and 9-packs should be chosen? Surprisingly, it does not matter! Whichever of the 4 we choose will allow us to find all remaining factorizations of 120 (and of any other McNugget number, for that matter).

As an example, suppose we choose the relation $(10, 0, 0) \sim (0, 0, 3)$. Starting with the initial factorization $(10, 0, 3)$, we can trade 6's for 20's to obtain $(0, 0, 6)$. Moreover, we can instead trade 20's for 6's in $(10, 0, 3)$ and obtain

$$(20, 0, 0),\ (17, 2, 0),\ (14, 4, 0),\ (11, 6, 0),\ (8, 8, 0),\ (5, 10, 0),\ \text{and}\ (2, 12, 0)$$

by subsequently trading 6's for 9's using $(3, 0, 0) \sim (0, 2, 0)$. These turn out to be the final factorizations of $120 \in$ 🐔. Had we instead chosen to use the relation $(4, 4, 0) \sim (0, 0, 3)$, we can still obtain the factorization $(0, 0, 6)$, this time starting with $(4, 4, 3)$ and trading all of the 6's and 9's for 20's, and the remaining factorizations in the centered expression above can be obtained by swapping out the 20's in $(10, 0, 3)$, and then once again repeatedly applying $(3, 0, 0) \sim (0, 2, 0)$.

Figure 3. The 20-piece box.

It can be helpful to record the above information using diagrams like the ones in Figure 4 (called *factorization graphs*). Both graphs depict all of the factorizations of $120 \in$ 🐔, but the left hand graph connects any two vertices with an edge if they are related by a single trade of $(3, 0, 0) \sim (0, 2, 0)$ or $(10, 0, 0) \sim (0, 0, 3)$, while the right hand graph uses the relations $(3, 0, 0) \sim (0, 2, 0)$ and $(4, 4, 0) \sim (0, 0, 3)$. In both examples, we began at a factorization in the middle column, and used our second relation to branch out to the remaining columns. Note that the edges of the factorization graph depend on a particular choice of relations.

This illustrates the concept of a *minimal presentation*, which is a collection ρ of relations such that for an element n, any two factorizations of n are connected by a sequence of trades using only the relations in ρ. Said another way, a collection of relations forms a minimal presentation if for every n, the factorization graph whose edges come from ρ is connected.

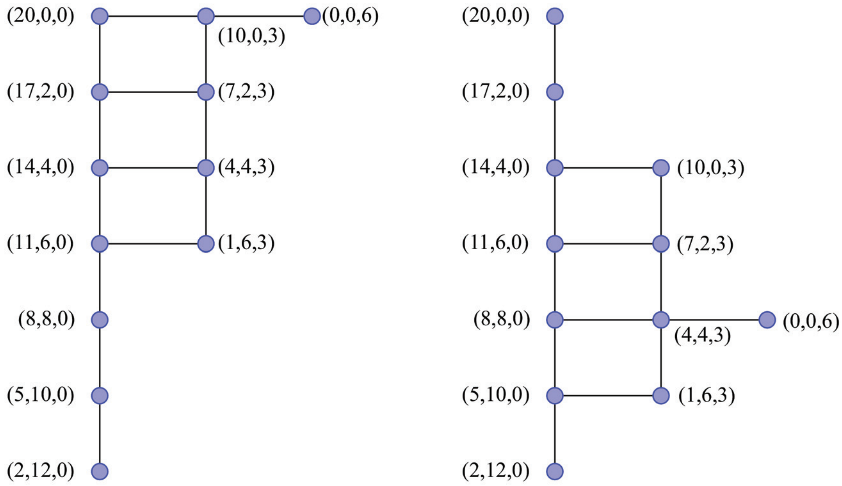

Figure 4. The factorization graph of 120 in the Chicken McNugget Monoid with two different choices of minimal relations.

The word "minimal" here means that none of the relations are implied by the rest (for instance, the relation $(6, 0, 0) \sim (0, 4, 0)$ would be redundant in 🐔 since it can be obtained by applying $(3, 0, 0) \sim (0, 2, 0)$ twice).

Much is known about the structure of minimal presentations. For instance, all of the minimal presentations for a given numerical monoid $\langle n_1, \ldots, n_k \rangle$ will have the same number of relations, and the elements whose factorizations appear in these relations will be identical as well. Indeed, all 4 minimal presentations for 🐔 involve one relation between factorizations of 18 and one relation between factorizations of 60. One way to see this is that if we tried to build a minimal presentation ρ using only the relation $(3, 0, 0) \sim (0, 2, 0)$, then 60 would be the smallest element whose factorization graph was disconnected. This implies we must include in ρ some relation between factorizations of 60 to ensure that its factorization graph is connected. From there, as noted above, no matter which relation we pick, the factorization graphs of all remaining elements of 🐔 will be connected. Indeed, this characterizes minimal presentations; they are minimal sets of relations so that any factorization graph is connected (see [25, Chapter 7] for thorough and precise definitions).

Throughout the remainder of this paper, we will use minimal presentations and factorization graphs to develop new invariants, which will measure the relationships between the atoms of a general numerical monoid. This will be completely analogous to how the elasiticy and delta set invariants measure the size and complexity of factorization lengths. Along the way, we will encouter more graphs related to the factorization graph, but all will be different in key ways. All of these graphs will be vital in our eventual arguments.

The amazing distance function

In the previous section, we saw the role that trades play in the structure of a numerical monoid $\langle n_1, \ldots, n_k \rangle$. In order to define invariants from this structure, we need a way to measure which trades are "larger" than others. Under such a measure, a trade $z_1 \sim z_2$ should be smaller than one which involves more atoms changing hands. Before continuing, we need to define what "more" means.

Given the important role that factorization lengths have played, it is tempting to consider the difference $\big||z_1| - |z_2|\big|$ in factorization lengths between z_1 and z_2 as a possible measure. However, this has a drawback; consider the element $n = 126 \in$ , which has factorizations

$$\mathsf{Z}(126) = \{(21,0,0), (18,2,0), (15,4,0), (12,6,0), (9,8,0), (6,10,0),$$
$$(3,12,0), (0,14,0), (11,0,3), (8,2,3), (5,4,3), (2,6,3), (1,0,6)\}.$$

Lurking in this set of factorizations is the trade $(11,0,3) \sim (0,14,0)$, which has a length difference of 0, despite 14 atoms being passed in each direction! Clearly, this will not do.

With this in mind, we consider the following measure of the "size" of a trade, one which focuses on the maximum length attained by the trade factorizations instead of their length difference. We will make this definition in general terms so that it applies to all numerical monoids, and follow up with a concrete example.

Definition 2. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid with $n \in S$. Suppose $z_1 = (x_1, \ldots, x_k)$ and $z_2 = (y_1, \ldots, y_k)$ are both in $\mathsf{Z}(n)$ and set

$$z_1 \wedge z_2 = (\min(x_1, y_1), \ldots, \min(x_n, y_n)).$$

The *distance* between the two factorizations z_1 and z_2 of n is given by

$$\mathbf{d}(z_1, z_2) = \max\{|z_1|, |z_2|\} - |z_1 \wedge z_2| \in \mathbb{N}_0.$$

If $n = 126$, $z_1 = (0,14,0)$, $z_2 = (11,0,3)$, and $z_3 = (3,12,0)$, then we have

$$\mathbf{d}(z_1, z_2) = 14 - 0 = 14,$$
$$\mathbf{d}(z_2, z_3) = 15 - 3 = 12,$$
$$\mathbf{d}(z_1, z_3) = 15 - 12 = 3.$$

Intuitively, $\mathbf{d}(z_1, z_2)$ equals the maximum length of z_1 and z_2 where we have ignored the atoms appearing in both z_1 and z_2. This ensures that a trade such as $(2,6,3) \sim (2,6,3)$ has distance 0, which is reasonable considering that applying this trade has no net effect on the starting factorization.

The distance function is an example of a *metric*, meaning that it satisfies many of the same basic properties that other distances function do (you may have encountered metrics in an analysis class). We gather some facts below and encourage the reader to work out their proofs as an exercise (the interested reader can also consult [**20**, Proposition 1.2.5] for arguments).

Proposition 3. *If $S = \langle n_1, \ldots, n_k \rangle$ is a numerical monoid and $n \in S$ with $z_1, z_2, z_3 \in \mathsf{Z}(n)$, then we have the following:*

1. $\mathbf{d}(z_1, z_2) = 0$ *if and only if* $z_1 = z_2$;
2. *if* $z_1 \neq z_2$, *then* $2 \le \mathbf{d}(z_1, z_2) \le \mathrm{L}(n) < \infty$;
3. $\mathbf{d}(z_1, z_2) = \mathbf{d}(z_2, z_1)$; *and*
4. $\mathbf{d}(z_1, z_2) \le \mathbf{d}(z_1, z_3) + \mathbf{d}(z_2, z_3)$.

The final item in Proposition 3 is known as the *triangle inequality*, which, broadly speaking, ensures that one cannot find a strictly shorter distance between two points by first traveling to a third.

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	(9, 1, 2)	(6, 3, 2)	(3, 5, 2)	(0, 7, 2)
(9, 1, 2)	0	3	6	9
(6, 3, 2)		0	3	6
(3, 5, 2)			0	3
(0, 7, 2)				0

Table 1. Distances between McNugget expansions of 103.

We conclude this section with one last example, which will be used in the following section. In Table 1, we compute all the possible distances between factorizations of $103 \in$ 🐔. Due to Proposition 3.3, we need only fill in the top half of the table.

On telephone poles and chains of factorizations

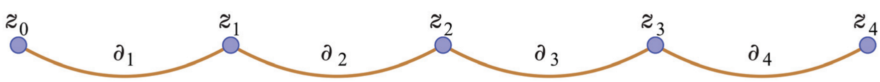
So now that we know how to measure distances between factorizations, let us apply this to create an invariant that describes the distribution of the distances in $\mathsf{Z}(n)$.

Definition 4. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$. A sequence of factorizations

$$z_0, z_1, \ldots, z_t$$

where each $z_i \in \mathsf{Z}(n)$ is called a *chain of factorizations* of n. For each $i \in \{1, \ldots, t\}$, set $\eth_i = \mathbf{d}(z_{i-1}, z_i)$ which we refer to as the *length* of the i-th link of the chain.

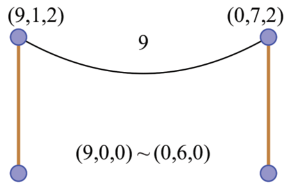
Thus, in one sense you can think of a chain of the form z_0, z_1, z_2, z_3, z_4 in terms of the following picture, where the $\eth_i$'s represent the lengths of each individual "link" in the chain. You can even think of the $\eth_i$'s as "weights" of the links.



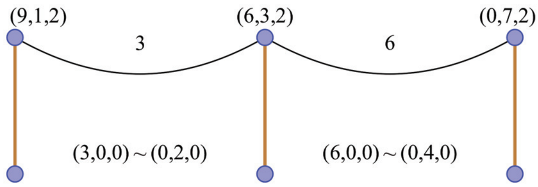
Given any two factorizations z and z' of an element $n \in \langle n_1, \ldots, n_k \rangle$, one can build infinitely many chains between them, since in the definition of chain there is no stipulation that the z_i's need be distinct. We in some sense want to find a chain linking z and z' that uses links of minimal distance. Hence, we introduce the following definition.

Definition 5. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$, and let N be a positive integer. A chain of elements $z_0, z_1, \ldots, z_t$ in $\mathsf{Z}(n)$ is called an N-chain if each distance $\eth_i \leq N$ for $i \in \{1, \ldots, t\}$.

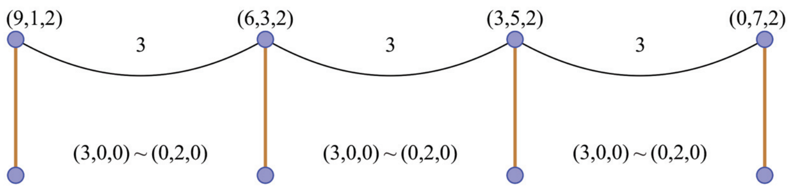
We extend the picture from above to provide examples of N-chains. The chain image has been extended to a sequence of "telephone poles" labeled at the top with particular factorizations, and at the bottom by the trades being performed. We use factorizations of $103 \in$ 🐔 from Table 1. The following depicts a 9-chain from (9, 1, 2) to (0, 7, 2).



By substituting the link for one that passes through (6, 3, 2), we obtain a 6-chain.



By substituting one more time the link between (6, 3, 2) and (0, 7, 2) for one through (3, 5, 2), we can reduce further to a 3-chain.



As such, even though (9, 1, 2) and (0, 7, 2) are distance 9 apart, we can obtain one from the other using only trades with distance at most 3.

Distances required to build chains: the catenary degree

In the way that elasticity analyzes the "spread" of factorization lengths of an element, and the delta set analyzes the relative distribution within the set of lengths, how might one use the distance function to describe the structure of the set $\mathsf{Z}(n)$? The answer lies in the N-chains constructed above. In the case where factorizations in $\mathsf{Z}(n)$ are in close proximity to each other, one would expect to be able to construct an N-chain between any two factorizations for a small value of N. The larger this necessary value of N, the more complex the structure of $\mathsf{Z}(n)$. This motivates the following definition.

Definition 6. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$. The *catenary degree* of n is defined as

$$\mathrm{c}(n) = \min\{N : \text{there exists an } N\text{-chain beween any } z_1, z_2 \in \mathsf{Z}(n)\}.$$

We define the *catenary degree* of S to be

$$\mathrm{c}(S) = \sup\{\mathrm{c}(n) : n \in S\}.$$

Before continuing, we note that the study of the catenary degree in numerical monoids has been a frequent topic in the recent mathematical literature [**1**, **6**, **9**, **10**, **23**]. To understand some of the intricacies involved in studying the catenary degree, we will need to consider some of its elementary properties.

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Figure 5. Telephone poles never looked so good.

Since the distance function cannot equal 1, $\mathsf{c}(n) = 0$ if and only if $|\mathsf{Z}(n)| = 1$ (that is, n has unique factorization) and thus $\mathsf{c}(n) \geq 2$ if and only if $|\mathsf{Z}(n)| > 1$. Moreover, it is easy to argue that $|\mathsf{Z}(n)| < \infty$ for every $n \in S$. Thus the set

$$D(n) = \{\mathbf{d}(z_1, z_2) : z_1, z_2 \in \mathsf{Z}(n)\}$$

is finite. If $M > \max D(n)$, and z_1 and $z_2 \in \mathsf{Z}(n)$, then any chain from z_1 to z_2 is an M-chain, and hence $\mathsf{c}(n) < \infty$. We summarize these fundamental observations in the next result.

Proposition 7. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$.*

1. $\mathsf{c}(n) = 0$ *if and only if* $|\mathsf{Z}(n)| = 1$.
2. *If* $|\mathsf{Z}(n)| \neq 1$, *then* $2 \leq \mathsf{c}(n) < \infty$.
3. *Hence,* $\mathsf{c}(S) = 0$ *or* $2 \leq \mathsf{c}(S)$.

It turns out that $\mathsf{c}(S)$ is always finite (and hence equal to the maximum of the catenary degrees achieved by the elements of S), though we defer a discussion on this matter until after the introduction of the tame degree in the next section.

While many of the references cited above work on computations of $\mathsf{c}(S)$, there is a relatively simple algorithm for obtaining $\mathsf{c}(n)$ from the set $\mathsf{Z}(n)$ using a graph similar to those used earlier. Given $n \in S$, let $\mathcal{D}_n$ denote the complete graph whose vertices are the elements of $\mathsf{Z}(n)$, and label the edge between the factorizations z_1 and z_2 with $\mathbf{d}(z_1, z_2)$. We will refer to $\mathcal{D}_n$ as the *distance graph* of n with respect to S.

Example 8. Consider the distance graph of $103 \in$ 🐔, depicted in Figure 6. One way to obtain the catenary degree is to remove edges from $\mathcal{D}_{103}$, starting with those of highest weight, until removing a particular edge disconnects the graph (such edges are known as *bridges*). The weight of the last edge removed equals the catenary degree. Several of the graphs resulting from this process are depicted alongside the full distance graph in Figure 6. Removing any one edge would disconnect the last graph, so the catenary degree of 103 is $\mathsf{c}(103) = 3$. Our implementation is essentially the well-known "reverse-delete" algorithm which first appeared in a paper by Kruskal [22].

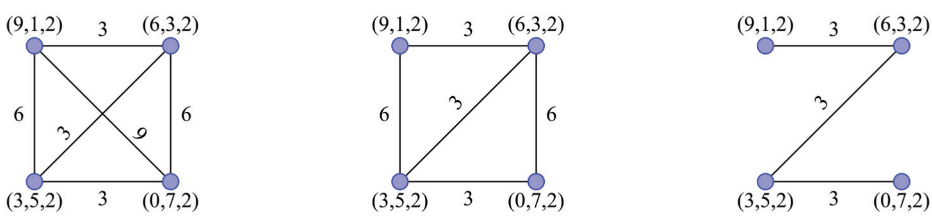


Figure 6. The distance graph of 103 in the Chicken McNugget monoid, in full (left) and with some edges removed (middle and right). As removing any remaining edge would yield a disconnected graph, $\mathsf{c}(103) = 3$.

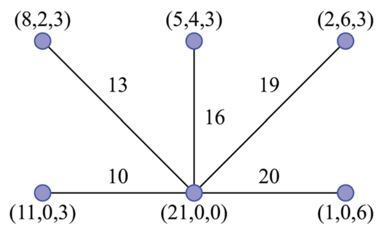
Before proceeding on to a definition of the tame degree, we outline in less formal language the meaning of the catenary degree.

Summary: Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$.

1. $\mathrm{c}(n) = N$ means that N is the smallest positive integer such that an N-chain exists between any two factorizations of n.
2. $\mathrm{c}(S) = N$ means that N is the smallest positive integer such that given any element $m \in S$, an N-chain exists between any two factorizations of m.

Distances required to reach atoms: the tame degree

While the catenary degree measures length in terms of chains, the tame degree measures distance from factorizations containing a specified atom. To motivate this invariant, we return to $126 \in$ 🐓 and note the set $\mathsf{Z}(126)$ computed earlier. Notice that $(21, 0, 0) \in \mathsf{Z}(126)$ and does not contain any copies of the atom 20. How close is it to a factorization that does? There are 5 such factorizations, and we list their distances from $(21, 0, 0)$ in the following diagram.



So $(21, 0, 0)$ is at minimum 10 units distance from any factorization of 126 which contains a copy of 20. We invite the reader to repeat this process on the remaining 7 factorizations of 126 which do not contain a copy of 20; you will find that each such factorization is 10 units (or less) away from a factorization with a copy of 20.

Measuring minimal distances from an arbitrary factorization to one that contains a specific atom is the idea behind the tame degree. We give the technical definition of the tame degree below.

Definition 9. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid with $n \in S$.

1. For each i with $n - n_i \in S$, denote by $\mathrm{t}(n, n_i)$ the minimum t such that for every $z \in Z(n)$, there exists a factorization $z' \in \mathsf{Z}(n)$ with $z' = (y_1, \ldots, y_n)$ where $y_i \neq 0$ and $\mathbf{d}(z, z') \leq t$. If $n - n_i \notin S$, then define $\mathrm{t}(n, n_i) = 0$.
2. The *tame degree* of n is $\mathrm{t}(n) = \max\{\mathrm{t}(n, n_i) : 1 \leq i \leq k\}$.
3. The *tame degree* of S is $\mathrm{t}(S) = \sup\{\mathrm{t}(n) : n \in S\}$.

Hence, to compute $\mathrm{t}(n, n_i)$, for every factorization in $\mathcal{D}_n$ where the n_i-th coordinate is zero, we compute the minimum distance to a factorization where that coordinate is nonzero. Thus, returning to $126 \in$ 🐓, our previous work has shown that $\mathrm{t}(126, 20) = 10$. Notice that this required 40 distance calculations. In a similar fashion, we obtain $\mathrm{t}(126, 6) = 3$ and $\mathrm{t}(126, 9) = 7$, meaning

$$\mathrm{t}(126) = \max\{10, 3, 7\} = 10.$$

How is one to interpret this? Given any factorizaton $z \in \mathsf{Z}(126)$, you can “tame” (or “keep apart”) any two factorizations of 126 containing an arbitrarily chosen atom with a *whip* of length 10.

We establish some elementary properties of the tame degree in the next proposition, and as earlier leave the proofs to the reader.

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Proposition 10. *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$.*

1. *We have* $\mathrm{t}(n) = 0$ *if and only if all factorizations in* $\mathsf{Z}(n)$ *have identical support.*
2. *We have* $\mathrm{t}(n) \leq \mathrm{L}(n) < \infty$. *Hence either* $\mathrm{t}(n) = 0$ *or* $2 \leq \mathrm{t}(n) < \infty$.

While we have shown above a simple algorithm using graphs to compute $\mathrm{c}(n)$ for $n \in S$, we note that the computation of $\mathrm{t}(n)$ is in general much more complicated and not as intuitive. Hence, we close this section with a summary of the various tame degree definitions in practical terms.

Summary: let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and $n \in S$.

1. $\mathrm{t}(n, n_i) = m$ means m is the smallest nonnegative integer such that if $z \in \mathsf{Z}(n)$, then there is some factorization $z' \in \mathsf{Z}(n)$ containing at least one copy of n_i that is within distance m of z.
2. $\mathrm{t}(n) = m$ means m is the smallest nonnegative integer such that if $z \in \mathsf{Z}(n)$, then for each $i \in \{1, \ldots, k\}$, there is some factorization $z' \in \mathsf{Z}(n)$ containing at least one copy of n_i that is within m units of z.
3. $\mathrm{t}(S) = m$ means m is the smallest nonnegative integer such that if $n \in S$ and $z \in \mathsf{Z}(n)$, then for each $i \in \{1, \ldots, k\}$, there is some factorization $z' \in \mathsf{Z}(n)$ containing at least one copy of n_i that is within m units of z.

Computing catenary and tame degrees of a numerical monoid

While we have argued in Propositions 7 and 10 that $\mathrm{c}(n)$ and $\mathrm{t}(n)$ are always finite, we have skirted the larger issue of the finiteness of $\mathrm{c}(S)$ and $\mathrm{t}(S)$. To settle this point, we appeal to the following result proven by undergraduates in an NSF-supported REU program from the summer of 2013.

Theorem 11 ([9, Theorem 3.1]). *Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical monoid and suppose that $L = \mathrm{lcm}\{n_1, \ldots, n_k\}$. The sequences $\{\mathrm{c}(n)\}_{n \in S}$ and $\{\mathrm{t}(n)\}_{n \in S}$ are eventually periodic with fundamental period a divisor of L.*

Thus, if m is the point in S at which $\{\mathrm{c}(n)\}_{n \in S}$ becomes periodic, then

$$\mathrm{c}(S) \in \{\mathrm{c}(n) : n \in S,\ n \leq m + L\}$$

and hence must be finite. Similar reasoning holds for $\mathrm{t}(S)$.

Corollary 12. *If $S = \langle n_1, \ldots, n_k \rangle$ is a numerical monoid, then both $\mathrm{c}(S)$ and $\mathrm{t}(S)$ are finite.*

Example 13. The catenary degrees of the elements of $\langle 5, 11, 12 \rangle$ are depicted in Figure 7. One can readily observe that for $n \geq 55$, the catenary degree $\mathrm{c}(n)$ is periodic in n with fundamental period 5.

Corollary 12 reduces the computation of $\mathrm{c}(S)$ and $\mathrm{t}(S)$ to a finite set of elements. With a little more work we can do even better, restricting to so-called *Betti elements* for the catenary degree and the *Apéry set* for the tame degree. In the remainder of this section, we explore these constructions.

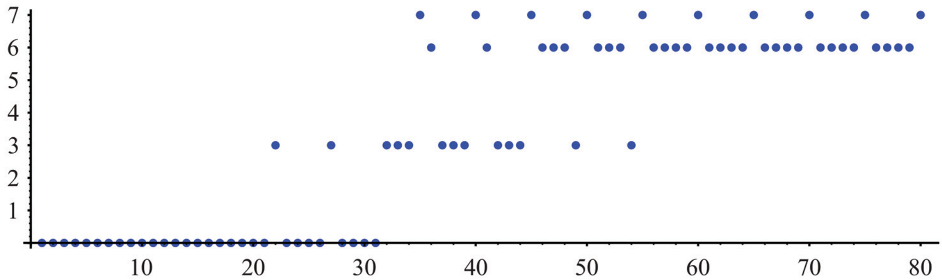

Figure 7. A plot in which each point (n, N) indicates $\mathsf{c}(n) = N$ for $n \in \langle 5, 11, 12\rangle$.

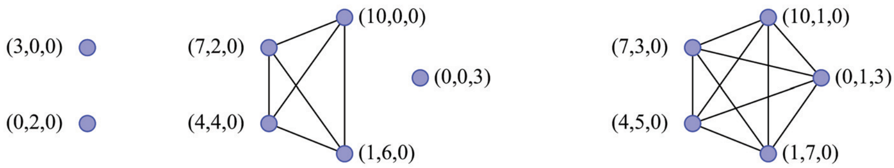

Figure 8. The Betti graphs of 18 (left), 60 (center), and 69 (right) in the Chicken McNugget Monoid.

Those beautiful Betti elements and awesome Apéry sets Let us return to the idea of *minimal presentations* from earlier. As we saw, given a numerical monoid $S = \langle n_1, \ldots, n_k\rangle$, a minimal presentation is a set of trades with which, for any $n \in S$, one can obtain any factorization in $\mathsf{Z}(n)$ from any other. Using the language of chains, if N is the highest trade distance in a minimal presentation of S, then there exists an N-chain between any two factorizations of n.

This allows us to identify which elements of S are key to computing $\mathsf{c}(S)$.

Definition 14. Let $S = \langle n_1, \ldots, n_k\rangle$ be a numerical monoid. For $n \in S$, construct a graph $\mathcal{G}_n$, called the *Betti graph*, whose vertices are the factorizations in $\mathsf{Z}(n)$, where an edge between z_1 and z_2 is included if z_1 and z_2 have at least one atom in common. We call n a *Betti element* of S if the Betti graph of n is not connected. Denote by $\mathrm{Betti}(S)$ the set of Betti elements of S.

Returning to 🐔, we see in Figure 8 that $\mathcal{G}_{18}$ consists of two vertices and no edges, and $\mathcal{G}_{60}$ has two connected components, one consisting of all factorizations involving 6's and 9's and the other a factorization using 20's. Disconnected Betti graphs indicate that any minimal presentation must necessarily include a trade bridging the connected components. On the other hand, Figure 8 demonstrates that $\mathcal{G}_{69}$ is connected so 69 is not a Betti element. As it turns out, Betti(🐔) $= \{18, 60\}$.

In order to locate the Betti elements of $S = \langle n_1, \ldots, n_k\rangle$, we need to introduce a certain finite set of elements that sits at the heart of numerical monoids. For motivation, consider the elements of 🐔 when organized based on their equivalence class modulo 6:

$$🐔 = \left\{\begin{matrix} \mathbf{0}, & 6, & 12, & \ldots, & \mathbf{49}, & 55, & 61, & \ldots, & \mathbf{20}, & 26, & 32, & \ldots, \\ \mathbf{9}, & 15, & 21, & \ldots, & \mathbf{40}, & 46, & 52, & \ldots, & \mathbf{29}, & 35, & 41, & \ldots \end{matrix}\right\}.$$

Since 🐔 is closed under addition, every element of 🐔 can be obtained by adding a multiple of 6 to one of the bolded values above, each of which is the smallest element of 🐔 in its equivalence class modulo 6. This leads to the following crucial definition.

Definition 15. Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical monoid. For a nonzero $n \in S$, the *Apéry set* of n in S, is defined and denoted as

$$\mathrm{Ap}(S, n) = \{s \in S : s - n \notin S\}.$$

As discussed above, it is easy to see that there is a unique element in $\mathrm{Ap}(S, n)$ for each congruence class modulo n, each of which is precisely the minimum element of S in its congruence class modulo n. In particular, $|\,\mathrm{Ap}(S, n)| = n$.

Let us examine why Apéry sets arise in the computation of the Betti elements. Assume that you have a bunch of factorizations, e.g., the factorizations

$$\mathsf{Z}(60) = \{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$$

of 60. In order to move from $(10, 0, 0)$ to $(7, 2, 0)$, we remove their "common part" $(7, 0, 0)$ and apply the trade $(3, 0, 0) \sim (0, 2, 0)$. Observe that $(3, 0, 0)$ and $(0, 2, 0)$ are factorizations of the same element $60 - (7 \cdot 6) = 18 \in$ 🐔, and the factorizations

$$\mathsf{Z}(18) = \{(3, 0, 0), (0, 2, 0)\}$$

of 18 have no common part, so there is no common part to remove. This means that the factorization where 6 appears has no atom in common with any other factorizations (in this case, only $(0, 2, 0)$). Hence $18 - 9$ is in 🐔, because we have a factorization where 9 occurs, but $(18 - 9) - 6$ cannot be in 🐔, since this would imply that there is a factorization of 18 where 6 and 9 both occur. This means $18 - 9 \in \mathrm{Ap}($🐔$, 6)$, and $18 = n_i + w$ for $w = 18 - 9$ and $i \neq 1$.

Notice that if we want to go from $(10, 0, 0)$ to $(4, 4, 0)$, the common part is $(4, 0, 0)$ and the new "bridge" is $6 \cdot 6 = 4 \cdot 9 = 36$, with factorizations

$$\mathsf{Z}(36) = \{(6, 0, 0), (3, 2, 0), (0, 4, 0)\}.$$

Since we want to move from $(6, 0, 0)$ to $(0, 4, 0)$, we can use the fact that $(3, 2, 0)$ shares 3 copies of 6 with $(6, 0, 0)$ and 2 copies of 9 with $(0, 4, 0)$. In both situations, the problem reduces to moving from $(3, 0, 0)$ to $(0, 2, 0)$, which was already considered above as the factorizations of a Betti element. We can argue analogously with the rest of factorizations of 60 that share some atoms, but there is one specific factorization, $(0, 0, 3)$, that does not share atoms with the rest. Since we can move freely now with trades in $\{(10, 0, 0), (7, 2, 0), (1, 6, 0), (4, 4, 0)\}$, it suffices to add a new trade to go from this set to $(0, 0, 3)$ (thus the different possible choices for minimal presentations for 🐔). Observe that in this case there is no factorization containing both 6 and 20. This means that $60 - 20 \in$ 🐔 but $(60 - 20) - 6 \notin$ 🐔, and as above $60 = (60 - 20) + 20$, with $60 - 20 \in \mathrm{Ap}($🐔$, 6)$ and 20 a generator other than 6. This idea is behind the following result, which we will later find very useful.

Theorem 16. *[3, Proposition 49] Let $S = \langle n_1, \dots, n_k \rangle$ be a numerical monoid minimally generated by $n_1, \dots, n_k$ where $n_1 < n_2 < \cdots < n_k$. If s is a Betti element of S, then $s = n_i + w$ where $i \in \{2, \dots, k\}$ and $w \in \mathrm{Ap}(S, n_1)\backslash\{0\}$.*

We note that the converse of Theorem 16 is false; elements of the form $s = n_i + w$ in the theorem must be filtered before determining if they yield Betti elements. By Theorem 16, the computation of $\mathrm{Betti}(S)$ for a given $S = \langle n_1, \dots, n_k \rangle$ is a finite process, but can be complicated, especially if k is relatively large. In fact, the size of $\mathrm{Betti}(S)$ can be arbitrarily large, even in the case $k = 4$ (in [8] a family with arbitrary number of Betti elements is given). We will address this issue later, but for now we show why we are so interested in Betti elements.

Theorem 17. *[**11**, Theorem 3.1] For any numerical monoid* $S = \langle n_1, \ldots, n_k \rangle$,

$$\mathrm{c}(S) = \max\{\mathrm{c}(n) : n \in \mathrm{Betti}(S)\}.$$

Example 18. We offer a very simple example to illustrate the ideas just presented. Let a and b be relatively prime positive integers with $1 < a < b$, and set $S = \langle a, b \rangle$. The elements of S are of the form $ax + by$ where x and y are nonnegative integers. Using Theorem 16, it is easy to reason that

$$\mathrm{Betti}(S) = \{ab\} \text{ and } \mathrm{Ap}(S, a) = \{0, b, 2b, \ldots, (a-1)b\}.$$

Indeed, $\mathsf{Z}(ab) = \{(b, 0), (0, a)\}$ and the Betti graph of any other element is either a single vertex (if $n - ab \notin S$) or connected (if $n - ab \in S$ is positive). Thus, $\mathrm{c}(\langle a, b \rangle) = \mathrm{c}(ab)$. Since $\mathbf{d}((b, 0), (0, a)) = b$, we conclude $\mathrm{c}(\langle a, b \rangle) = b$.

There is a somewhat similar method for computing $\mathrm{t}(S)$, though as with computing individual values of $\mathrm{t}(n)$, it is more expensive to complete. The method we will use centers around the following result.

Theorem 19. *[**10**, Theorem 16] Let* $S = \langle n_1, \ldots, n_k \rangle$ *where the generating set for* S *is minimal in cardinality. If* n *is minimal in* S *such that* $\mathrm{t}(n) = \mathrm{t}(S)$*, then* $n = w + n_i$ *for some* $i \in \{1, \ldots, k\}$ *and* $w \in \mathrm{Ap}(S, n_j)$ *with* $j \in \{1, \ldots, k\} \setminus \{i\}$.

We note that there is an alternate method to compute $\mathrm{t}(S)$ which involves the computation of the *primitive elements* of $\langle n_1, \ldots, n_k \rangle$, analogous to the Betti elements for the catenary degree; the interested reader should consult [**11**, Proposition 4.1].

Example 20. Returning to Example 18, we again have that $\mathrm{t}(\langle a, b \rangle) = \mathrm{t}(ab)$, and as such, since $\mathbf{d}((b, 0), (0, a)) = b$, we conclude $\mathrm{t}(\langle a, b \rangle) = b$.

We saw above that the Betti elements of S were enough to compute the catenary degree of a numerical monoid, and these could be computed from the minimal generators and an Apéry set. Thus computing the tame degree in general requires more machinery than computing the catenary degree.

Calculations for the Chicken McNugget Monoid

We begin with the Apéry set of $6 \in$ 🐔, which is

$$\mathrm{Ap}(S, 6) = \{0, 49, 20, 9, 40, 29\},$$

written so the i-th element is the minimum element in 🐔 congruent with i modulo 6. According to Theorem 16, the candidates for Betti elements are

$$\{18, 29, 38, 40, 49, 58, 60, 69\}.$$

We use GAP to find the factorizations of these elements, which are listed in Table 2.

In Figure 8, we have seen that $\mathcal{G}_{18}$ and $\mathcal{G}_{60}$ are disconnected and that $\mathcal{G}_{69}$ is connected. The Betti graphs of 29, 40, and 49 are trivially connected as each is uniquely factorable, and those of 38 and 58 are connected by the trade $(3, 0, 0) \sim (0, 2, 0)$ alone. Thus, the only Betti elements of 🐔 are 18 and 60. The since $\mathcal{G}_{18}$ consists of two vertices, $\mathrm{c}(18) = \max\{2, 3\} = 3$. We can compute the catenary degree of

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n	$\mathsf{Z}(n)$ in 🐔
18	$\{(3, 0, 0), (0, 2, 0)\}$
29	$\{(0, 1, 1)\}$
38	$\{(3, 0, 1), (0, 2, 1)\}$
40	$\{(0, 0, 2)\}$
49	$\{(0, 1, 2)\}$
58	$\{(3, 0, 2), (0, 2, 2)\}$
60	$\{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0), (0, 0, 3)\}$
69	$\{(10, 1, 0), (7, 3, 0), (4, 5, 0), (1, 7, 0), (0, 1, 3)\}$

Table 2. Factorizations of elements necessary to compute the catenary degree.

60 using the method outlined in Figure 6, and we reason through this procedure using relations. In order to move from any factorization to another in the set $\{(10, 0, 0), (7, 2, 0), (4, 4, 0), (1, 6, 0)\}$ we just need the relation $(3, 0, 0) \sim (0, 2, 0)$, which in terms of the catenary degree has a cost of three. And the shortest distance from this set to $(0, 0, 3)$ is attained by choosing $(1, 6, 0)$. This implies that $\mathrm{c}(60) = 7$ and hence by Theorem 17, we conclude c(🐔) = 7.

Now let us focus in the tame degree. According to Theorem 19 we need to consider the factorizations of the elements in n + Ap(🐔, m) for distinct $n, m \in \{6, 9, 20\}$. We already know Ap(🐔, 6); it is easy to check that

Ap(🐔, 10) $= \{0, 46, 20, 12, 40, 32, 6, 52, 26\}$,

and

Ap(🐔, 20) $= \{0, 21, 42, 63, 24, 45, 6, 27, 48, 9, 30, 51, 12, 33, 54, 15, 36, 57, 18, 39\}$.

So our set of elements of the form $n + w$ with n a minimal generator of 🐔 and w in the Apéry set of another minimal generator is

$$\{6, 9, 12, 15, 18, 20, 21, 24, 26, 27, 29, 30, 32, 33, 36, 38, 39,$$
$$40, 42, 45, 46, 48, 49, 51, 52, 54, 57, 58, 60, 63, 66, 69, 72\}.$$

Among these elements, 6, 9, 12, 15, 20, 21, 26, 29, 32, 40, 46, 49, and 52 each have a single factorization, and thus need not be considered. The factorizations of the remaining elements can each be found in Table 2 or 3.

Observe that the tame degrees of 33, 39, 51, and 57 are each zero by Proposition 10, since all of their factorizations involve only the first two generators. The maximum distance between factorizations for 18, 24, 27, 30, 38, 58 is three. Notice that in the expressions of 36, 42, 45, 48, 54, and 63, only the first two generators appear (hence, these are acting like factorizations in the numerical monoid $\langle 2, 3 \rangle$ and the tame degree of this monoid is 3; see Example 20). Thus, the tame degrees of 18, 24, 27, 30, 36, 38, 42, 45, 48, 54, 58, and 63 are all 3.

So it remains to see what the tame degrees of 60, 66, 69, and 72 are. We will only examine 60 here, as the remaining elements can be handled in a similar fashion. If we focus on the first generator, 6, which appears in $(10, 0, 0)$, we have to find the closest factorization where 6 does not occur. The only candidate is $(0, 0, 3)$, and $\mathbf{d}((10, 0, 0), (0, 0, 3)) = 10$. The distance between any other factorization where 6 is involved and $(0, 0, 3)$ (the only one where 6 does not occur) is less than 10. But these

n	$\mathsf{Z}(n)$ in 🐔
24	$\{(4,0,0),(1,2,0)\}$
27	$\{(3,1,0),(0,3,0)\}$
30	$\{(5,0,0),(2,2,0)\}$
33	$\{(4,1,0),(1,3,0)\}$
36	$\{(6,0,0),(3,2,0),(0,4,0)\}$
39	$\{(5,1,0),(2,3,0)\}$
42	$\{(7,0,0),(4,2,0),(1,4,0)\}$
45	$\{(6,1,0),(3,3,0),(0,5,0)\}$
48	$\{(8,0,0),(5,2,0),(2,4,0)\}$
51	$\{(7,1,0),(4,3,0),(1,5,0)\}$
54	$\{(9,0,0),(6,2,0),(3,4,0),(0,6,0)\}$
57	$\{(8,1,0),(5,3,0),(2,5,0)\}$
63	$\{(9,1,0),(6,3,0),(3,5,0),(0,7,0)\}$
66	$\{(11,0,0),(8,2,0),(5,4,0),(2,6,0),(1,0,3)\}$
69	$\{(10,1,0),(7,3,0),(4,5,0),(1,7,0),(0,1,3)\}$
72	$\{(12,0,0),(9,2,0),(6,4,0),(3,6,0),(0,8,0),(2,0,3)\}$

Table 3. Factorizations of elements necessary to compute the tame degree.

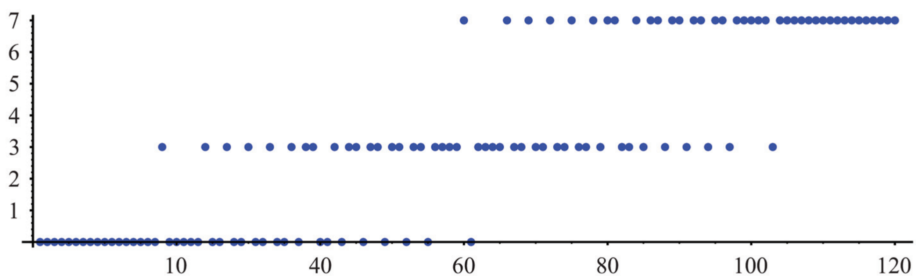

Figure 9. A plot in which each point (n, N) indicates $\mathsf{c}(n) = N$ for $n \in \langle 6, 9, 20\rangle$.

factorizations are precisely those where 9 appears, and so the tame degree does not grow when we look at the second generator. Now for the last generator, 20, the only factorization in which it appears is $(0, 0, 3)$, and the closest where 20 does not occur is $(1, 6, 0)$, and $\mathbf{d}((0, 0, 3), (1, 6, 0)) = 7$. It follows that $\mathsf{t}(60) = 10$, and one can show that the same holds for 66, 69 and 72.

In total, we have obtained the following.

Proposition 21. *We have* $\mathsf{c}($🐔$) = 7$ *and* $\mathsf{t}($🐔$) = 10$.

We close by returning to Theorem 11 and give a complete description of the periodic behavior of the sequences $\{\mathsf{c}(s)\}_{s\in}$🐔 and $\{\mathsf{t}(s)\}_{s\in}$🐔. Both sequences must have a fundamental period which divides $\operatorname{lcm}\{6, 9, 20\} = 180$, and Figures 9 and 10 give strong indication of the values indicated in Observation 22. However, no proof of these observations are known aside from carefully examining factorizations and making arguments for each equivalence class modulo the fundamental periods, a particularly arduous task for the tame degree with its period of 60.

Observation 22.

1. The sequence $\{\mathsf{c}(n)\}_{n\in}$🐔 has fundamental period 1, which begins at $n = 104$. Hence, for $n \geq 104$, $\mathsf{c}(n) = 7$.

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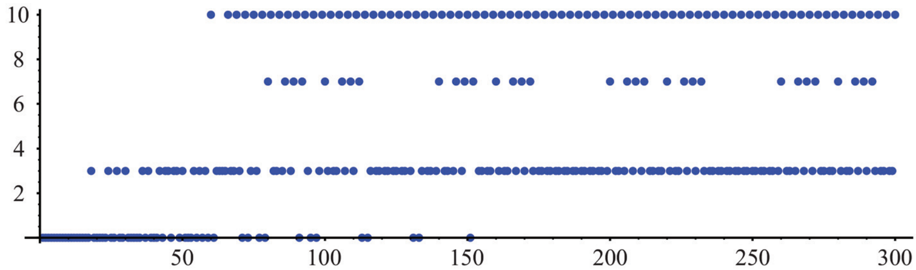

Figure 10. A plot in which each point (n, N) indicates $\mathrm{t}(n) = N$ for $n \in \langle 6, 9, 20\rangle$.

2. The sequence $\{\mathrm{t}(n)\}_{n\in \text{🐔}}$ has fundamental period 60, which begins at $n = 152$.

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Summary. We use the Chicken McNugget Monoid to demonstrate the various properties involving nonunique factorization relating to chains of factorizations. We study in depth the catenary and tame degrees of this monoid.

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