On Length Densities

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What is factorization theory?

A commutative cancellative monoid M with set of irreducible elements (or atoms) $\mathcal{A}(M)$ is called *atomic* if for each nonunit $x \in M$ there are $x_1, \ldots, x_k \in \mathcal{A}(M)$ such that $x = x_1 \cdots x_k$. For such an x, set

$$\mathcal{L}(x) = \{n \in \mathbb{N} \mid \text{there exists atoms } x_1, \ldots, x_k \text{ with } x = x_1 \cdots x_k\}. \quad (1)$$

The set $\mathcal{L}(x)$ is known as the *set of lengths* of $x \in M$, and its study over the past 60 years has been the principal focus of non-unique factorization theory.

In particular, much of this work has centered on combinatorial constants related to $\mathcal{L}(x)$.

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Introduction

For instance, if M is the multiplicative monoid of an integral domain R then set

$$L(x) = \max \mathcal{L}(x),\ \ell(x) = \min \mathcal{L}(x),\ \rho(x) = \frac{L(x)}{\ell(x)},$$

$$\text{and } \rho(M) = \inf\{\rho(x) \mid x \in M\}.$$

The constant $\rho(x)$ is known as the *elasticity* of x in M and the constant $\rho(M)$ as the *elasticity of* M.

Further set

$$\overline{L}(x) = \lim_{n\to\infty} \frac{L(x^n)}{n} \text{ and } \overline{\ell}(x) = \lim_{n\to\infty} \frac{\ell(x^n)}{n}. \quad (2)$$

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Anderson and Pruis show in PAMS **113**(1991) that

(i) both the limits $\overline{L}(x)$ and $\overline{\ell}(x)$ exist (although $\overline{L}(x)$ may be infinite);

(ii) if α and $\beta \in [0, \infty]$ with $0 \leq \alpha \leq 1 \leq \beta \leq \infty$, then there is an integral domain R and an irreducible $x \in R$ with $\overline{\ell}(x) = \alpha$ and $\overline{L}(x) = \beta$.

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Introduction

Definition

Let M be a commutative cancellative atomic BF-monoid with set of units $M^\times$. Define a function $L^\Delta : M \to \mathbb{N}_0$ via

$$L^\Delta(x) = \max \mathcal{L}(x) - \min \mathcal{L}(x)$$

where we define $L^\Delta(x) = 0$ if $x \in M^\times$. We define the *length kernel* of M, denoted M^{LK}, as the kernel of L^Δ and the *length ideal* of M, denoted M^{LI}, as $M \setminus M^{LK}$. For $x \in M^{LI}$ set

$$\mathrm{LD}(x) = \frac{|\mathcal{L}(x)|-1}{L^\Delta(x)},$$

which we call the *length density* of x.

Definition

Moreover, set
$$\mathrm{LD}(M) = \inf\{\mathrm{LD}(x) \mid x \in M^{LI}\},$$
which we call the *length density* of M. If there is an $x \in M^{LI}$ such that $\mathrm{LD}(M) = \mathrm{LD}(x)$, then we say that the length density of M is *accepted*. Set
$$\overline{\mathrm{LD}}(x) = \lim_{n\to\infty} \mathrm{LD}(x^n)$$
to be the *asymptotic length density of* x, provided this limit exists.

Basic Ideas and Bounds on the Length Density

We open by considering the largest value that $\mathrm{LD}(x)$ can attain.

Proposition

Let M be a commutative cancellative atomic monoid and $x \in M^{LI}$. The following statements are equivalent.

1. $\mathrm{LD}(x) = 1$.
2. $\mathcal{L}(x)$ *is an interval.*
3. $\Delta(x) = \{1\}$.

If the elements of M satisfy any of these conditions, then it necessarily has accepted length density.

Comment: While many such M can be constructed (using numerical and congruence monoids), there does not seem to be an indepth study of such monoids in the factorization literature.

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Basic Ideas and Bounds on the Length Density

We start with a more general proposition.

Proposition

Let $x \in M^{LI}$. *Then*

$$\frac{1}{\max \Delta(x)} \leq LD(x) \leq \frac{1}{\min \Delta(x)}, \tag{3}$$

with equality if and only if $|\Delta(x)| = 1$. *Hence*

$$\frac{1}{\max \Delta(M)} \leq LD(M) \leq \frac{1}{\min \Delta(M)}. \tag{4}$$

Immediately we obtain the following.

Corollary

Let M be an atomic monoid. If $\Delta(M) = \{d\}$ for some positive integer d, then $\mathrm{LD}(x) = \frac{1}{d}$ and consequently $\overline{\mathrm{LD}}(x) = \frac{1}{d}$ for all $x \in M^{LI}$. It follows that $\mathrm{LD}(M) = \frac{1}{d}$ and that the length density of M is accepted.

Example

The bounds in the last Proposition may be strict. For instance, take the numerical moniod $M = \langle 6, 9, 20 \rangle$. Here $\Delta(M) = \{1, 2, 3, 4\}$, and we have $LD(18) = 1$ and $LD(60) = \frac{4}{7}$. It can be shown that $LD(M) = \frac{4}{7}$, and that $LD(x) \to 1$ as $x \to \infty$. For example, $LD(1000) = \frac{109}{112}$.

Examples

Example

We construct a monoid with nonzero rational length density which is not accepted. Let M be the quotient of the free abelian monoid on atoms $a_1, a_2, \ldots$ with the minimal relations

$$a_1^3 = a_2^4 = a_3^6$$
$$a_4^3 = a_5^4 = a_6^6 = a_7^8$$
$$a_8^3 = a_9^4 = a_{10}^6 = a_{11}^8 = a_{12}^{10}$$
$$a_{13}^3 = a_{14}^4 = a_{15}^6 = a_{16}^8 = a_{17}^{10} = a_{18}^{12}$$
$$\vdots$$

Examples

Example

Note that every atom is contained in exactly one minimal relation and hence, this monoid is an FF-monoid. Further, $\Delta(x) = \{1, 2\}$ for every element of M with nonunique factorization, and $\rho(M) = \infty$.
Consider the elements $b_1, b_2, b_3, \ldots$ defined by the minimal relations, i.e. $b_1 = a_1^3, b_2 = a_4^3, \ldots$. We have

$$\mathcal{L}(b_i) = \{3, 4, 6, \ldots, 2(i+2)\}$$

and from this

$$LD(b_i) = \frac{i+1}{2i+1}.$$

Taking $i \to \infty$, we get $\frac{1}{2}$, and therefore $LD(M) \leq \frac{1}{2}$. But also $LD(M) \geq \frac{1}{2}$, as $2 = \max \Delta(M)$. Hence $LD(M) = \frac{1}{2}$. It cannot be accepted, because no $x \in M$ has $\Delta(x) = \{2\}$.

The Other Extreme

We now work toward the other extreme and start with a definition.

Definition

Let M be an atomic monoid. If for every nonempty finite subset $S \subset \{2, 3, 4, \ldots\} = \mathbb{N} - \{1\}$ there exists an element $x \in M^{LI}$ such that $\mathcal{L}(x) = S$, then we say that M has the *Kainrath Property*.

Clearly a monoid M with the Kainrath property satisfies $\Delta(M) = \mathbb{N}$. We deduce another Corollary.

Corollary

If M has the Kainrath property, then $\mathrm{LD}(M) = 0$ and $\{\mathrm{LD}(x) \,|\, x \in M^{LI}\} = (0, 1]$. Hence, a monoid with the Kainrath property does not have accepted length density.

Nonrational and Accepted Elasticity

A fundamental question early in the study of elasticity was whether or not an integral domain can have irrational elasticity.

Let $a, b \in \mathbb{N}$ with $b > a$. Let $c \in [0, 1]$. For each $i \in \mathbb{N}$, set $k(i) = \lceil ic(b-a) \rceil$. We will now define the monoid $M(a, b, c)$, as the free abelian monoid on atoms $\{q_{i,j} : i, j \in \mathbb{N}\}$, with minimal relations:

$$\forall i \in \mathbb{N}, \qquad q_{i,ia}^{ia} = q_{i,ia+1}^{ia+1} = q_{i,ia+2}^{ia+2} = \cdots = q_{i,ia+k(i)}^{ia+k(i)} = q_{i,ib}^{ib}.$$

Proposition

Let $a, b \in \mathbb{N}$ with $b > a$. Let $c \in [0, 1]$. Then $\rho(M(a, b, c)) = \frac{b}{a}$ and $LD(M(a, b, c)) = c$.

Finitely Generated Monoids have Accepted Length Density

Theorem

If S is a finitely generated semigroup, then $\mathrm{LD}(S)$ is accepted.

Proof.

Involves an examination of the Betti elements of S. □

Block Monoids

We briefly approach the question of computing $\mathrm{LD}(\mathcal{B}(G))$ and start with a known result concerning the delta set of such a block monoid. If $G = \sum_{i=1}^{k} \mathbb{Z}_{n_i}$ is a finite abelian group where $n_i | n_{i+1}$ for $1 \leq i < k$ with $|G| \geq 3$, then

$$[1, n_k - 2] \subseteq \Delta(\mathcal{B}(G)) \subseteq [1, \mathsf{c}(\mathcal{B}(G)) - 2] \subseteq [1, \mathsf{D}(G) - 2].$$

Here $\mathsf{D}(G)$ represents the *Davenport's Constant* of G. The quantity $\mathsf{c}(M)$ is the *catenary degree* of the monoid M.

Proposition

If G is a finite abelian group with $|G| \geq 3$, then

$$\frac{1}{\mathsf{c}(\mathcal{B}(G)) - 2} \leq \mathrm{LD}(\mathcal{B}(G)) \leq 1.$$

Block Monoids

Corollary

If $G = \mathbb{Z}_n$ *is cyclic, then* $\mathrm{LD}(\mathcal{B}(\mathbb{Z}_n)) = \dfrac{1}{n-2}$ *and if* $G = \sum_{i=1}^{k} \mathbb{Z}_2$*, then*

$$\mathrm{LD}(\mathcal{B}(\sum_{i=1}^{k} \mathbb{Z}_2)) = \frac{1}{k-1}.$$

We list an application of this Corollary to algebraic rings of integers.

Corollary

If R *is a ring of algebraic integers with class number* p *where* p *is prime, then* $\mathrm{LD}(R) = \dfrac{1}{p-2}$.

Asymptotic Length Density

We exhibit an atomic monoid with an element that lacks asymptotic length density.

Example

Consider the Puiseux monoid

$$M = \left\langle \frac{4}{3}, \frac{8}{5}, \frac{800}{1201}, \frac{a_1}{p_1}, \frac{a_2}{p_2}, \ldots \right\rangle .$$

The p_i are a strictly increasing sequence of primes, and the a_i are a strictly increasing sequence of natural numbers defined recursively. Using known results on Puiseux monoids, M is a BF-monoid (and an FF-monoid). Our focus is on $x = 8$, and we calculate $x^n = 8n$, as n grows large. An extended argument shows the following.

Example

Example

- For $n < 100$, $x^n < 800$, $LD(x^n) = 1$.
- At $n = 100$, $LD(x^{100}) < \frac{1}{2}$.
- As n continues to increase, so long as $8n < a_1$, all factorizations of x^n will include only the first three atoms. An extended computation shows that $LD(x^{2900}) > \frac{3}{4}$.
- We are now ready to choose the next atom. Set $a_1 = 2901 \cdot 8$, and $p_1 > 2 \cdot 1201 \cdot 30$, e.g. $p_1 = 72073$. Using only the first three atoms, all factorizations of x^{2901} are of length at most $30 \cdot 1201$. Using the new fourth atom, we get a new factorization of length p_1, and we can argue that $LD(x^{2901}) < \frac{1}{2}$.
- By continuing in this way, we find that $LD(x^n)$ can be made to grow to be above $\frac{3}{4}$, then to shrink below $\frac{1}{2}$, over and over as $n \to \infty$. Hence the asymptotic length density of x does not exist.

When do we get Asymptotic Length Densitites?

What atomic monoids admit asymptotic length densities for all their elements?

While we do not completely answer this, we offer a large class that does. If M is a monoid and $x \in M$, then let $\|x\|$ denote the set of all elements in M that divide x^k for some $k \in \mathbb{N}$. For $a \in M$ and $x \in Z(M)$, let $\mathrm{t}(a, x) \in \mathbb{N}_0 - \{1\}$ denote the the *tame degree* of a with respect to x. If $u \in M$, then set $\mathrm{t}(M, u) = \sup\{\mathrm{t}(x, u) \mid x \in M\}$.

The monoid M is *locally tame* if $\mathrm{t}(M, u) < \infty$ for each atom u of M. If M is an atomic locally tame monoid, then M is a BF-monoid. The *tame degree* of M, is defined by $\mathrm{t}(M) = \sup\{\mathrm{t}(M, u) \mid u \in \mathcal{A}(M)\}$. If $\mathrm{t}(M) < \infty$, then M is called *globally tame*. Since $\mathrm{c}(H) \leq \mathrm{t}(H)$, global tameness implies finiteness of the catenary degree.

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The Theorem

Theorem

Let S be a locally tame atomic monoid and for $x \in S$ set $H = \|x\|$. Assume for $x \in S$ that $\Delta(x) \neq \emptyset$ and $|\Delta(H)| < \infty$. Let $d = \min \Delta(H)$, τ be minimal such that $d \in \Delta(x^\tau)$, $\psi = \max(\tau, \rho(\Delta(H)) - 1)$, and $T = \mathrm{t}(H, Z(x^\psi))$. For all $n \geq \psi$ it follows that

$$\frac{1}{d} - \frac{2T}{nd^2} \leq LD(x^n) \leq \frac{1}{d}.$$

In particular, $\overline{LD}(x) = 1/d$.

Corollary

If S is NICE, then all elements of S admit asymptotic length densities.

Examples

Example

1. Finitely generated monoids are globally tame, hence all nonunit elements admit asymptotic length densities.
2. Let H be a Krull monoid with class group G and let $G_0 \subset G$ denote the set of classes containing prime divisors. If the Davenport constant $\mathsf{D}(G_0) < \infty$ (which holds if G_0 is finite), then H is globally tame. Thus such Krull monoids, such as the ring of algebraic integers in a finite extension of the rationals, satisfy Theorem 18.
3. Every C-monoid (see Definition 2.9.5 in GHKB) is locally tame and has finite catenary degree. Orders in algebraic number fields fall into this class.