Factorization in Leamer Monoids: ω-Primality in Arithmetic Leamer Monoids

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The Student Authors

This talk is based on the work completed under my direction at the 2013 PURE REU at the University of Hawaii at Hilo and the 2014 Sam Houston State REU by the following students.

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Zack Tripp

Two faculty also contributed to these results.

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Why Leamer Monoids?

Conjecture (Huneke-Wiegand)

Let R be a one-dimensional Gorenstein domain. Let $M \neq 0$ be a finitely generated R-module, which is not projective. Then the torsion submodule of $M \otimes_R \mathrm{Hom}_R(M, R)$ is non-trivial.

Proposition (García-Sánchez and Leamer, *J. Algebra* 2013)

Let Γ be a numerical monoid and $\mathbb{K}$ be a field. The monoid algebra $\mathbb{K}[\Gamma]$ satisfies the Huneke-Wiegand conjecture for monomial ideals generated by two elements if and only if for each $s \in \mathbb{N} \setminus \Gamma$, there exists an irreducible arithmetic sequence of the form $\{x, x+s, x+2s\}$ in Γ.

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Introduction

Definition

Given a numerical monoid Γ, $s \in \mathbb{N} \setminus \Gamma$, define

$$S_\Gamma^s = \{(0,0)\} \cup \{(x,n) : \{x, x+s, x+2s, \ldots, x+ns\} \subset \Gamma\} \subset \mathbb{N}^2.$$

That is, S_Γ^s is the collection of arithmetic sequences of step size s contained in Γ.

Example

For $\Gamma = \langle 7, 10 \rangle$ and $s = 3$, the graph of the Leamer monoid $S^3_{\langle 7,10 \rangle}$ is given below in Figure 2. In this example, Γ is generated by an arithmetic sequence with step size 3, which is equal to s. The Frobenius number of Γ (53 in this case) can be easily read off the graph as the rightmost column absent of any dots.

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The Notation of Factorization Theory

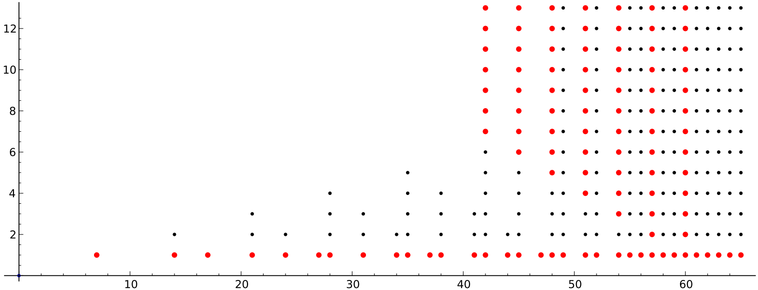

Figure: The Leamer Monoid $S^3_{\langle 7,10 \rangle}$.

Another Pictorial View

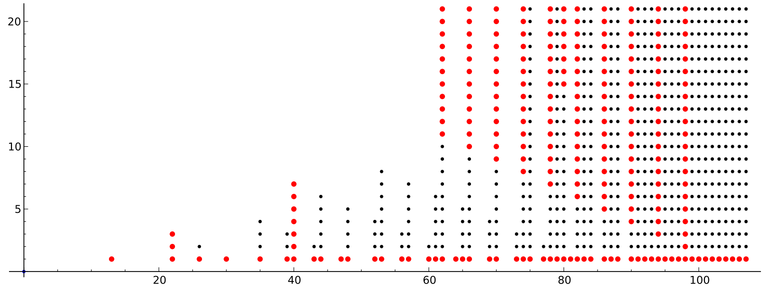

Figure: The Leamer Monoid $S^4_{\langle 13,17,22,40 \rangle}$.

Even More Notation

Consider factorizations of the form

$$x = x_1 \cdots x_k = y_1 \cdots y_t$$

which may not be unique.

If $x \in M^\bullet$, then *the set of lengths of* x is

$$\mathcal{L}(x) = \{k \in \mathbb{N} \mid x = a_1 a_2 \cdots a_k \text{ where } a_i \in \mathcal{A}(M(a,n))\}.$$

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The Notation Continues Coming

Set

$$\ell(x) = \min \mathcal{L}(x) \text{ and } L(x) = \max \mathcal{L}(x).$$

If $\mathcal{L}(x) = \{n_1, \ldots, n_t\}$ with the n_i's listed in increasing order, then set

$$\Delta(x) = \{n_i - n_{i-1} \mid 2 \leq i \leq t\}$$

and

$$\Delta(M) = \bigcup_{1 \neq x \in M^\bullet} \Delta(x).$$

If $\Delta(M) \neq \emptyset$, then,

$$\min \Delta(M) = \gcd \Delta(M).$$

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The Elasticity of Factorization

The elasticity of an element $x \in M^{\bullet}$, denoted $\rho(x)$, is given by

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More Definitions

For $z = (z_1, \ldots, z_p), z' = (z'_1, \ldots, z'_p) \in \mathbb{N}^p$ write

$$\gcd(z, z') = (\min\{z_1, z'_1\}, \ldots, \min\{z_p, z'_p\}),$$

and

$$\frac{z}{z'} = z - z'.$$

Define

$$\mathrm{d}(z, z') = \max\left\{\left|\frac{z}{\gcd(z,z')}\right|, \left|\frac{z'}{\gcd(z,z')}\right|\right\},$$

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Given $n \in S$ and $z, z' \in \mathbb{F}(n)$, an *N-chain of factorizations* from z to z' is a sequence $z_0, \ldots, z_k \in \mathbb{F}(n)$ such that $z_0 = z$, $z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq N$ for all i.

The *catenary degree* of n, $\mathrm{c}(n)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $z, z' \in \mathbb{F}(n)$, there is an N-chain from z to z'.

The catenary degree of S, $\mathrm{c}(S)$, is defined by

$$\mathrm{c}(S) = \sup\{\mathrm{c}(n) \mid n \in S\}.$$

Note: If S does not have unique factorization, then $\mathrm{c}(S) \geq 2$ and if $\mathrm{c}(S) = 2$, then S is half-factorial.

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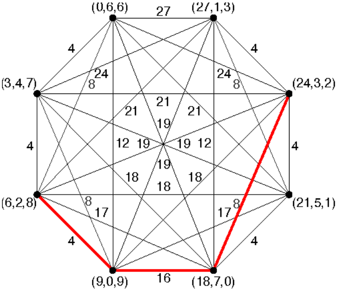
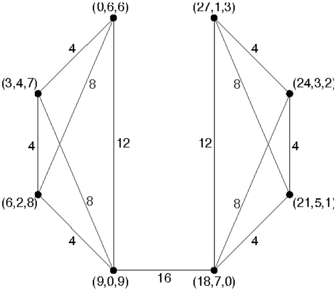
Pictorial View





$S = \langle 11, 36, 39 \rangle$ and $n = 450 \in S$

A Summary: Part I

	$\Gamma = \langle n_1, \cdots, n_k \rangle$	$\Gamma = \langle n, n+k, \ldots, n+tk \rangle$
ρ	$\rho(\Gamma) = \frac{n_k}{n_1}$	$\rho(\Gamma) = \frac{n+tk}{n}$
$\mathcal{L}$	Modified Kainrath	Arithmetic Sequences
Δ	Eventually Periodic	$\Delta(\Gamma) = \{k\}$
c	Eventually Periodic	$\mathrm{c}(\Gamma) = \lceil \frac{n}{t} \rceil + k$
ω	Eventually Quasi-Linear	Eventually Quasi-Linear

Notation

Definition

For a Leamer monoid S_Γ^s and $x \in \Gamma$, the *column at* x is the set

$$\{(x, n) \in S_\Gamma^s : n \geq 1\}.$$

If this set is empty, we say that no column exists at x. If a column exists at x, the column at x is said to be *finite* (resp., infinite) if the column has finite (resp., infinite) cardinality. The *height* of the finite column at x is

$$\max\{n : (x, n) \in S_\Gamma^s\}.$$

Definition

Given a Leamer monoid S_Γ^s, we use $x_0(S_\Gamma^s)$ to denote the smallest x such that $(x, 1) \in S_\Gamma^s$. We denote by $x_f(S_\Gamma^s)$ the first infinite column of S_Γ^s, that is, the smallest x such that $(x, n) \in S_\Gamma^s$ for all $n \geq 1$.

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Lemma

Lemma

Let S_Γ^s be a Leamer monoid.

- (a) *If $(x, 1) \in S_\Gamma^s$, then $(x, 1) \in \mathcal{A}(S_\Gamma^s)$.*
- (b) *For $n \gg 0$, $(x_f, n) \in \mathcal{A}(S_\Gamma^s)$.*
- (c) *If $(x, n) \in S_\Gamma^s$, then $(x, n') \in S_\Gamma^s$ for all $1 \le n' \le n$.*
- (d) *If $(x, n-1) \in \mathcal{A}(S_\Gamma^s)$ and $(x, n) \in S_\Gamma^s$ for $n > 2$, then $(x, n) \in \mathcal{A}(S_\Gamma^s)$.*
- (e) *If $(x, n-1) \in \mathcal{A}(S_\Gamma^s)$ and $(x - s, n) \in S_\Gamma^s$ for $n > 2$, then $(x - s, n) \in \mathcal{A}(S_\Gamma^s)$.*
- (f) *The column at every $x > \mathcal{F}(\Gamma)$ is infinite.*
- (g) *For all $x > \mathcal{F}(\Gamma) + x_0$ and $n \ge 2$, (x, n) is a reducible element in S_Γ^s.*

Elasticity

Theorem

For any Leamer monoid S_Γ^s, $\rho(S_\Gamma^s) = \infty$.

Theorem

Fix a Leamer monoid S_Γ^s. Let

$$C = \{(x, n) \in S_\Gamma^s \setminus \mathcal{A}(S_\Gamma^s) : (x, n+1) \notin S_\Gamma^s \setminus \mathcal{A}(S_\Gamma^s)\}$$

that is, the set of reducible elements which lie in finite or mixed columns and have maximal height. Then $|\Delta(S_\Gamma^s)| < \infty$, and in fact $\max \Delta(S_\Gamma^s) \leq n^* - 1$, where

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Definition

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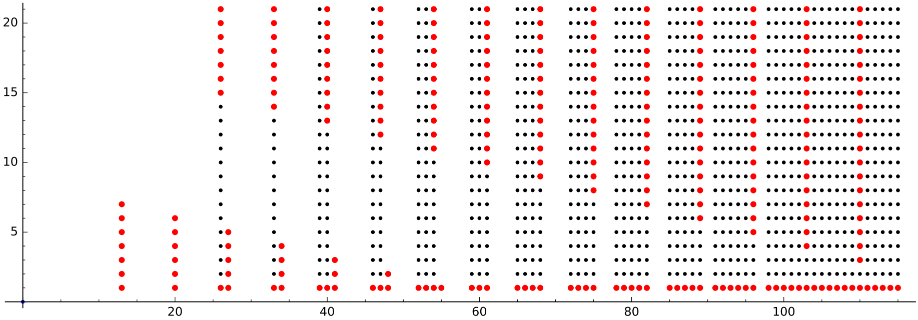
The Leamer monoid S^s_Γ is *arithmetical* if $\Gamma = \langle m, m+s, \ldots, m+ks \rangle$ for some $m, k \in \mathbb{N}$. For $m, k, s \in \mathbb{N}$ satisfying $1 \leq k \leq m-1$ and $\gcd(m, s) = 1$, let $\Gamma(m, k, s) = \langle m, m+s, \ldots, m+ks \rangle$, and let $S^s_{m,k} = S^s_{\Gamma(m,k,s)}$.

Graphical Example



The Leamer monoid $S_\Gamma^7 = S_{13,7}^7$ for $\Gamma = \langle 13, 20, 27, 34, 41, 48, 55, 62 \rangle$

First Results

Theorem

Fix an arithmetical Leamer monoid $S^s_{m,k}$. Then

$$\Delta(S^s_{m,k}) = \{1, \ldots, \lfloor \frac{m-2}{k} \rfloor + 1\}.$$

Corollary

The arithmetical Leamer monoid $S^s_{m,k}$ has catenary degree $\lfloor \frac{m-2}{k} \rfloor + 3$.

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Omega Function

Definition

Let S be a commutative cancellative monoid. For any nonunit $x \in S$, define $\omega(x) = m$ if m is the smallest positive integer such that whenever x divides $x_1 \cdots x_t$, with $x_i \in S$, then there is a set $T \subset \{1, 2, \ldots, t\}$ of indices with $|T| \leq m$ such that x divides $\sum_{i \in T} x_i$. If no such m exists, then set $\omega(x) = \infty$.

Definition

A product of irreducibles $x_1 \cdots x_k$ is said to be a bullet for n if n divides the product $x_1 x_2 \cdots x_k$ but does not divide any proper subproduct.

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More

Proposition

If M is a commutative cancellative monoid and x a nonunit of M, then

$$\omega(x) = \sup\{r \mid x_1 \cdots x_r \in \mathrm{bul}(x) \text{ where each } x_i \text{ is irreducible in } M\}.$$

Proposition

If $(x, n) \in S_\Gamma^s$, then (x, n) has a bullet of length $n+1$. Hence, $\omega((x, n)) \geq n+1$ and no element in a Leamer monoid is prime.

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Omega Formula for Arithemetical Leamer Monoids

Theorem

Let $S^d_{a,k}$ be an arithmetical Leamer monoid with $k \geq 2$.

1. *If $(x, n) \in S^d_{a,k}$ such that $(x, n) \neq p(a, k)$ for any $p \in \mathbb{N}$, then* $\omega((x, n)) = max\left(n + 1, m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} s \rfloor\right)$.
2. *If $(x, n) \in S^d_{a,k}$ such that $(x, n) = p(a, k)$ for some $p \in \mathbb{N}$, then* $\omega((x, n)) = n + 1$.

Omega Formula for Arithemetical Leamer Monoids

Theorem

Let $S_{a,k}^d$ be an arithmetical Leamer monoid with $k \geq 2$.

1. *If $(x, n) \in S_{a,k}^d$ such that $(x, n) \neq p(a, k)$ for any $p \in \mathbb{N}$, then* $\omega((x, n)) = max\left(n + 1, m + \lfloor \frac{a-2}{k} \rfloor + 1 + \lfloor \frac{a+i-1}{a} s \rfloor\right)$.
2. *If $(x, n) \in S_{a,k}^d$ such that $(x, n) = p(a, k)$ for some $p \in \mathbb{N}$, then* $\omega((x, n)) = n + 1$.

A Summary: Part II

	$\Gamma = \langle n_1, \cdots, n_k \rangle : S_\Gamma^s$	$\Gamma = \langle m, m+s, \ldots, m+ks \rangle : S_{m,k}^s$
ρ	$\rho(S_\Gamma^s) = \infty$	$\rho(S_{m,k}^s) = \infty$
$\mathcal{L}$	Hortizonal and Vertical Stability	Horizontal and Vertical Stability
Δ	$\lvert\Delta(S_\Gamma^s)\rvert < \infty$	$\Delta(S_{m,k}^s) = \{1, \ldots, \lfloor \frac{m-2}{k} \rfloor + 3\}$
c	$\mathrm{c}(S_\Gamma^s) < \infty$	$\mathrm{c}(\Gamma) = \lceil \frac{m-2}{k} \rceil + 3$
ω	$\omega((x, n)) \geq n + 1$	See Last Theorem!