# When Is A Puiseux Monoid Atomic?

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May 27, 2021

# References

This talk is based on several papers.


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2. F. Gotti, Increasing positive monoids of ordered fields are FF-monoids, *J. Algebra* **518**(2019), 40–56.
3. F. Gotti, On the atomic structure of Puiseux monoids, *J. Algebra Appl.* **16**(2017)), 1750126.
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# Motivation

Taken from: P. M. Cohen, Bezout rings and their subrings, *Proc. Cambridge Phil. Soc.* **64**(1968), 251-264.

An element of an integral domain is called an *atom* if it is a non-unit which cannot be written as a product of two non-units. If every element of a ring $R$ which is not a unit or 0 can be written as a product of atoms, $R$ is said to be *atomic*. The following result is easily verified:

PROPOSITION 1·1. *An integral domain is atomic if and only if it satisfies the maximum condition on principal ideals.*

# Motivation

We note that Cohen's maximal condition on principal ideals is equivalent to the acsending chain condition on principal ideals (a.c.c.p.).

Proposition $1 \cdot 1$ is incorrect. While Cohen is correct that $(\Leftarrow)$ is trivial to verify, $(\Rightarrow)$ fails.

The first example of an atomic integral domain without the a.c.c.p. appears in a paper by Anne Grams (Atomic rings and the ascending chain condition for principal ideals, *Math. Proc. Cambridge Philos. Soc.* **75**(1974), 321–329).

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# Grams' Example

Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be the ordered list of odd prime numbers. Let $\mathbb{F}$ be any field and $M$ be the additive submonoid of $\mathbb{Q}_{\geq 0}$, the nonnegative rationals, generated by

$$\left\{\frac{1}{2^n p_n}\right\}_{n\in\mathbb{N}}.$$

Let $D = \mathbb{F}[X; M]$ be the monoid ring over the field $\mathbb{F}$. If $S := \{f \in \mathbb{F}[X; M] : f(0) \neq 0\}$ is a multiplicatively closed subset of $D$, then the Grams' Example is

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# Three Comments About $M$

**Three Comments About** $M$. Set $\mathcal{B} = \left\{ \frac{1}{2^n p_n} \right\}_{n \in \mathbb{N}}$.

- Each $\frac{1}{2^m p_m}$ in $\mathcal{B}$ cannot be written as a linear combination over $\mathbb{N}_0$ of the remaining elements in $\mathcal{B}$. Thus, each element in $\mathcal{B}$ is an **atom** of $M$.
- As $M$ is generated by its atoms, $M$ **is atomic**.
- The monoid ideals

$$\frac{1}{2} + M \subsetneq \frac{1}{2^2} + M \subsetneq \frac{1}{2^3} + M \subsetneq \cdots$$

form an infinite ascending chain of ideals. **Hence** $M$ **does not satisfy the a.c.c.p.**

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# Definition

**In Grams' proof, the factorization properties of $M$ play a key role.**

**Definition**

A **Puiseux monoid** is an additive submonoid of $\mathbb{Q}_{\geq 0}$.

An interesting observation.

**Proposition**

*There are uncountably many non-isomorphic Puiseux monoids.*

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# Factorization Properties

The usual definitions involving factorization in commutative rings carry over to monoids in a natural manner.

### Definition

Let $M$ be a commutative cancellative monoid.

- A nonunit $x \in M$ is **irreducible** (or an **atom**) if $x = yz$ implies that $y$ or $z$ is a unit of $M$. We set $\mathcal{A}(M)$ to be the set of atoms of $M$.
- $M$ is **atomc** if every nonunit of $M$ can be written as a product of elements from $\mathcal{A}(M)$.
- $M$ is **antimatter** if $\mathcal{A}(M) = \emptyset$.

# Not Everything is Atomic

Unlike $M$ in the Grams' example, not all Puiseux monoids are atomic.

### Example

Let $p$ be a prime number and $M$ be the Puiseux monoid generated by the terms

$$\left\{\frac{1}{p^k}\right\}_{k\in\mathbb{N}}.$$

In such cases, we use the notation $\left\langle \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots \right\rangle$. Here $M$ is **not atomic**, as no generator is an atom. Here $\frac{1}{p^k} = p \cdot \frac{1}{p^{k+1}}$. In fact, $M$ has **no atoms**. We call such a Puiseux monoid an **antimatter** monoid.

# Question

**Question A:** When is a Puiseux monoid atomic?

**Question B:** When does a Puiseux monoid satisfy the a.c.c.p.?

**Question C:** When is a Puiseux monoid antimatter?

**Question D:** Given an atomic Puiseux monoid, what can we say about its factorization properties?

We focus on Question A, and give some scattered results along the way to Questions B, C, D.

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# A Simplier Case

The submonoids $S \subseteq \mathbb{N}$ under $+$, known as **numerical monoids** have been studied more carefully in these regards.

| Numerical Monoids | Puiseux Monoids |
|---|---|
| Always finitely generated | May not be finitely generated |
| Always atomic | May not be atomic |
| Always satisfies a.c.c.p. | May not satisfiy a.c.c.p. |
| Never antimatter | May be antimatter |
| Factorization properties well investigated | Factorization properties not well known |

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# When are they the same?

It is easy to determine which Puiseux monoids behave like numerical monoids.

**Proposition**

*A Puiseux monoid is isomorphic to a numerical monoid if and only if it is finitely generated.*

Hence, if a Puiseux monoid $M$ is finitely generated, then

- it has a unique minimal set of generators;
- it is atomic; its atoms are precisely the unique minimal generating set;
- it satisfies the a.c.c.p.

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# An Extension

Hence, we focus on the non-finitely generated case.

We can actually extend the last observations beyond the finitely genetated case.

**Proposition**

*Let $M$ be a Puiseux monoid. The following conditions are equivalent.*

1. *$M$ contains a minimal set of generators.*
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3. *$M$ is atomic.*

# An Example

We illustrate the last Proposition with an example.

**Example**

We use a construction somewhat similar to the Grams' example. Let

$$S = \{\frac{1}{2^n} \mid n \in \mathbb{N}\} \text{ and } \mathcal{P} = \{p_1, p_2, \ldots\}$$

be the ordered list of odd prime numbers. Let $M$ be the Puiseux monoid generated by $S \cup \mathcal{P}$. We have

- each $\frac{1}{p_n}$ is an atom of $M$;
- no $\frac{1}{2^n}$ can be written as a sum of atoms.

Thus $M$ is a non-atomic monoid with infinitely many atoms.

# The non-finitely generated case

We make another fundamental observation.

**Proposition**

*Let $M$ be a Puiseux monoid. If 0 is not a limit point of $M$, then $M$ is atomic.*

The proof is straightforward. Assume $M$ is not atomic and construct a sequence of elements in $M$ which converges to zero.

We note that Grams' example shows that the converse above does not hold.

**When 0 is a limit point, there is not much that can be said in general - the atomicity of this subclass of PMs is very complex and mostly not understood. We illustate this with some additional examples.**

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# Cyclic Semirings

## Definition

For $r \in \mathbb{Q}_{>0}$, let

$$S_r = \langle r^n \mid n \in \mathbb{N} \rangle .$$

$S_r$ is the **cyclic semiring** generated by $r$. We consider only the additive structure of $S_r$.

We completely describe the atomic structure of the $S_r$ monoids. If $r = \frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$, then we set $\mathsf{n}(r) = a$ and $\mathsf{d}(r) = b$.

## Proposition

1. *If $r \geq 1$, then $S_r$ is atomic and either*
   - $r \in \mathbb{N}$ *and so* $S_r \cong \mathbb{N}_0$,
   - *or* $r \notin \mathbb{N}$ *and so* $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.
2. *If $r < 1$, then*
   - *either* $\mathsf{n}(r) = 1$ *and so* $S_r$ *is antimatter,*
   - *or* $\mathsf{n}(r) \neq 1$ *and* $S_r$ *is atomic with* $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

# On the a.c.c.p.

**Corollary**

*For each* $r \in \mathbb{Q} \cap (0,1)$ *with* $\mathsf{n}(r) \neq 1$*, the monoid* $S_r$ *is an atomic monoid that does not satisfy the a.c.c.p.*

**Proof.**

We consider the principal ideals $\mathsf{n}(r)r^n + S_r$ for each $n \in \mathbb{N}$. Since

$$\mathsf{n}(r)r^n = \mathsf{d}(r)r^{n+1} = (\mathsf{d}(r) - \mathsf{n}(r))r^{n+1} + \mathsf{n}(r)r^{n+1},$$

$\mathsf{n}(r)r^{n+1} \mid_{S_r} \mathsf{n}(r)r^n$ for every $n \in \mathbb{N}_0$. Therefore

$$\mathsf{n}(r)r^n + S_r \subsetneq \mathsf{n}(r)r^{n+1} + S_r$$

is an ascending chain of principal ideals which never stabilizes. □

# A comment on antimatter monoids

Given a Puiseux monoid $M$, set

$$\mathbf{gp}(M) = \{x - y \mid x, y \in M\}.$$

$\mathbf{gp}(M)$ is a subgroup of $\mathbb{Q}$. We further set

$$\widetilde{M} = \{x \in \mathbf{gp}(M) \mid nx \in M \text{ for some } n \in \mathbb{N}\}$$

to be the **root closure** of $M$.

**Lemma**

*Let $M$ be a Puiseux monoid and $n = \gcd\{\mathrm{n}(x) \mid 0 \neq x \in M\}$. Then*

$$\widetilde{M} = n\left\langle \frac{1}{\mathrm{d}(x)} \mid 0 \neq x \in M \right\rangle.$$

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# A comment on antimatter monoids

**Proposition**

*A Puiseux monoid $M$ is not finitely generated if and only if $\widetilde{M}$ is antimatter.*

**Example**

Let $\mathbb{P}$ be the set of prime numbers and set $M = \left\langle \frac{1}{p} \mid p \in \mathbb{P} \right\rangle$. As 0 is a limit point of $M$, $M$ is not finitely generated. Thus, by the Proposition, $\widetilde{M}$ is antimatter. Clearly $1 = \gcd\{\mathrm{n}(x) \mid 0 \neq x \in M\}$ and that $\{\mathrm{d}(x) \mid 0 \neq x \in M\}$ is the set of square-free positive integers. Thus

$$\widetilde{M} = \left\langle \frac{1}{n} \mid n \in \mathbb{N} \text{ is square-free} \right\rangle$$

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